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Abstract

This paper is concerned with the existence and qualitative properties of transition fronts for spatially periodic reaction-diffusion equations with bistable nonlinearities. The notion of transition fronts connecting two stable steady states generalizes the standard notion of pulsating fronts. In this paper, we prove that the time-global solutions in the class of transition fronts share some common features. In particular, we establish a uniform estimate for the mean speed of transition fronts, independently of the spatial scale. Under the a priori existence of a pulsating front with nonzero speed or under a more general condition guaranteeing the existence of such a pulsating front, we show that transition fronts are reduced to pulsating fronts, and thus are unique up to shift in time. On the other hand, when the spatial period is large, we also obtain the existence of a new type of transition fronts which are not pulsating fronts. This example, which is the first one in periodic media, shows that even in periodic media, the notion of generalized transition fronts is needed to describe the set of solutions connecting two stable steady states.

1 Introduction and main results

This paper is devoted to the study of existence and qualitative properties of generalized fronts of one-dimensional spatially periodic reaction-diffusion equations with bistable nonlinearities. It is a

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follow-up to the paper [16] which dealt with the case of pulsating fronts. In [16] and the present paper, we consider periodic reaction-diffusion equations of the type

$$u_t = (a_L(x)u_x)_x + f_L(x,u), \quad t \in \mathbb{R},\ x \in \mathbb{R} \tag{1.1}$$

with $L > 0$, where u_t stands for $u_t = \partial_t u(t,x) = \partial u/\partial t(t,x)$, u_x stands for $u_x = \partial_x u(t,x) = \partial u/\partial x(t,x)$ and $(a_L(x)u_x)_x$ stands for $(a_L(x)u_x)_x = \partial_x(a_L u_x)(t,x) = \partial(a_L \partial u/\partial x)/\partial x(t,x)$. The diffusion and reaction coefficients a_L and f_L are given by

$$a_L(x) = a\Big(\frac{x}{L}\Big) \quad \text{and} \quad f_L(x,u) = f\Big(\frac{x}{L}, u\Big),$$

where the function $a : \mathbb{R} \to \mathbb{R}$ is positive, of class $C^{1,\alpha}(\mathbb{R})$ (with $0 < \alpha < 1$) and 1-periodic, that is, $a(x+1) = a(x)$ for all $x \in \mathbb{R}$. Throughout the paper, the function $f : \mathbb{R} \times [0,1] \to \mathbb{R}$, $(x,u) \mapsto f(x,u)$ is continuous, 1-periodic in x, of class $C^{0,\alpha}$ in x uniformly in $u \in [0,1]$, and of class $C^{1,1}$ in u uniformly in $x \in \mathbb{R}$. One also assumes that 0 and 1 are uniformly (in x) stable zeroes of $f(x,\cdot)$, in the sense that there exist $\gamma > 0$ and $\delta \in (0,1/2)$ such that

$$\begin{cases} f(x,0) = f(x,1) = 0 & \text{for all } x \in \mathbb{R}, \\ f(x,u) \le -\gamma u & \text{for all } (x,u) \in \mathbb{R} \times [0,\delta], \\ f(x,u) \ge \gamma(1-u) & \text{for all } (x,u) \in \mathbb{R} \times [1-\delta,1]. \end{cases} \tag{1.2}$$

Notice that this implies in particular that $\max\big(\partial_u f(x,0), \partial_u f(x,1)\big) \le -\gamma$ for all $x \in \mathbb{R}$.

A particular case of such a function f satisfying (1.2) is the cubic nonlinearity

$$f(x,u) = u(1-u)(u-\theta_x), \tag{1.3}$$

where $x \mapsto \theta_x$ is a 1-periodic $C^{0,\alpha}(\mathbb{R})$ function ranging in $(0,1)$. Notice that in (1.3) the intermediate zero θ_x of $f(x,\cdot)$ is not assumed to be constant. More generally speaking, under the assumption (1.2), the function $f(x,\cdot)$ may have several zeroes in the open interval $(0,1)$ and these zeroes may not be constant or of constant number. However, in one result of the paper, namely Theorem 1.7 below, we will assume that for every $x \in \mathbb{R}$, the function $f(x,\cdot)$ has exactly one zero in the interval $(0,1)$.

The bistable equation (1.1) arises naturally in modeling a variety of physical and biological phenomena, such as phase field models of solidification, signal propagation along bistable transmission lines, propagation of nerve pulses and population biology, see, e.g., [2, 4, 5, 19, 41, 44, 55, 56, 58]. In particular, in the context of population dynamics, the quantity $u(t,x)$ represents the population density at location x and at time t, the coefficient $a_L(x)$ is the diffusion rate at location x and the reaction term $f_L(x,u)$ measures the growth rate of the population density u at location x. The dependency of $a_L(x)$ and $f_L(x,u)$ on x allows to model the effects of the features of the habitat on the population density. The periodicity condition on $a_L(x)$ and $f_L(x,u)$ with respect to x indicates a typical spatial heterogeneity of the habitat. The bistable assumption (1.2) means that the growth rate $f_L(x,u)$ is negative at low densities, which refers to a strong Allee effect [29, 42, 52].

In the sequel, for mathematical purposes, the function f is extended in $\mathbb{R} \times (\mathbb{R}\backslash[0,1])$ as follows: $f(x,u) = \partial_u f(x,0)u$ for $(x,u) \in \mathbb{R} \times (-\infty,0)$ and $f(x,u) = \partial_u f(x,1)(u-1)$ for $(x,u) \in \mathbb{R} \times (1,+\infty)$. Thus, f is continuous in $\mathbb{R} \times \mathbb{R}$, 1-periodic in x, $\min_{x\in\mathbb{R}} f(x,u) > 0$ for all $u < 0$ and $\max_{x\in\mathbb{R}} f(x,u) < 0$ for all $u > 1$, while $f(x,u)$ and $\partial_u f(x,u)$ are globally Lipschitz-continuous in u uniformly in $x \in \mathbb{R}$.

Transition fronts

The notion of transition fronts was introduced by Berestycki and Hamel [9, 10] to describe a general class of front-like solutions for reaction-diffusion equations in unstructured heterogeneous media. For one-dimensional equations such as (1.1), the notion of transition fronts can be presented precisely as follows (see also the notion of wave-like one-dimensional solutions defined by Shen [46]).

Definition 1.1. *For problem* (1.1)*, a transition front connecting* 0 *and* 1 *is a time-global solution* $u : \mathbb{R} \times \mathbb{R} \to [0, 1]$ *for which there exists a family* $(\xi_t)_{t\in\mathbb{R}}$ *of real numbers such that*

$$\begin{cases} u(t, x + \xi_t) \to 1 & \text{as } x \to -\infty, \\ u(t, x + \xi_t) \to 0 & \text{as } x \to +\infty, \end{cases} \quad \text{uniformly in } t \in \mathbb{R}. \tag{1.4}$$

Clearly, from the strong maximum principle, any transition front u connecting 0 and 1 for (1.1) is such that $0 < u(t, x) < 1$ for all $(t, x) \in \mathbb{R}^2$. We point out that the uniformity in t is essential in this definition. As a matter of fact, there are solutions satisfying the limits of (1.4) pointwise in $t \in \mathbb{R}$, but not uniformly. Consider for instance the case of (1.1) when both a and f do not depend on x (whence $\theta_x = \theta \in (0, 1)$ is independent of x): this homogeneous equation admits some time-global solutions u such that $0 < u(t, x) < 1$ in $\mathbb{R}^2$, $u(t, -\infty) = 1$ and $u(t, +\infty) = 0$ for every $t \in \mathbb{R}$, while $u(t, x)$ is close to θ on some unbounded intervals as $t \to -\infty$, see [34]. Thus, these solutions are not transition fronts connecting 0 and 1 in the sense of Definition 1.1.

For a given transition front u of problem (1.1), the real numbers ξ_t reflect the positions of u as time runs. These real numbers ξ_t are not uniquely defined since, for any bounded function $t \mapsto \overline{\xi}_t$, the family $(\xi_t+\overline{\xi}_t)_{t\in\mathbb{R}}$ is also associated with u in the sense of (1.4). However, it follows straightforwardly from Definition 1.1 that the family $(\xi_t)_{t\in\mathbb{R}}$ is unique up to an additive bounded function. Furthermore, the distance between ξ_t and any level set of $u(t, \cdot)$ is uniformly bounded in t: indeed, for any real numbers b_1 and b_2 with $0 < b_1 \le b_2 < 1$, there is a constant $C = C(u, b_1, b_2) \ge 0$ such that

$$\big\{x \in \mathbb{R} \ ; \ b_1 \le u(t, x) \le b_2\big\} \subset \big[\xi_t - C, \xi_t + C\big] \quad \text{for every} \quad t \in \mathbb{R}.$$

Thus, an equivalent definition to (1.4) is that a transition front connecting 0 and 1 is a time-global solution $u : \mathbb{R} \times \mathbb{R} \to [0, 1]$ that converges to the steady states 0 (on the right) and 1 (on the left) far away from any of its level sets, uniformly in t.

An important notion attached to a transition front is the limiting average speed, if any, of the distance between the positions ξ_t.

Definition 1.2. *We say that a transition front connecting* 0 *and* 1 *for* (1.1) *admits a global mean speed* $c \ge 0$ *if*

$$\frac{|\xi_t - \xi_s|}{|t - s|} \to c \ \text{ as } |t - s| \to +\infty.$$

For a given transition front u, the global mean speed, if it exists, is uniquely determined and does not depend on the special choice of the positions $(\xi_t)_{t\in\mathbb{R}}$, since they are defined up to a bounded function (see also [10, Theorem 1.7]). Thus, the notion of global mean speed is meaningful.

A simple example of a transition front connecting 0 and 1 for (1.1) is a pulsating front, that is, a global solution $u : \mathbb{R} \times \mathbb{R} \to [0, 1]$ such that there exist a real number c_L (the average speed) and a function $\phi : \mathbb{R} \times \mathbb{R} \to [0,1]$ satisfying

$$\begin{cases} u(t,x) = \phi(x - c_L t, x/L) \text{ for all } (t,x) \in \mathbb{R} \times \mathbb{R}, \\ \phi(\xi, y) \text{ is 1-periodic in } y, \\ \phi(-\infty, y) = 1, \ \phi(+\infty, y) = 0 \text{ uniformly in } y \in \mathbb{R}. \end{cases} \tag{1.5}$$

Clearly, any pulsating front with average speed $c_L \in \mathbb{R}$ is a transition front with positions $\xi_t = c_L t$ for all $t \in \mathbb{R}$, whence $|c_L|$ is the global mean speed. If $c_L \neq 0$, then the map $(t,x) \to (x - c_L t, x/L)$ is a bijection from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R} \times \mathbb{R}$ and ϕ is uniquely determined by u. In this case, (1.5) is equivalent to the following definition

$$\begin{cases} u(t + L/c_L, x) = u(t, x - L) \quad \text{for all } (t,x) \in \mathbb{R}^2, \\ u(t, -\infty) = 1, \quad u(t, +\infty) = 0 \quad \text{locally in } t \in \mathbb{R}. \end{cases} \tag{1.6}$$

As a matter of fact, definition (1.6) was first given in [49] to denote pulsating fronts with nonzero speed. On the other hand, a pulsating front with speed $c_L = 0$ simply means a steady solution $u(t,x) = \phi_0(x)$ of (1.1) such that $\phi_0 : \mathbb{R} \to [0,1]$, $\phi_0(-\infty) = 1$ and $\phi_0(+\infty) = 0$. Throughout this paper, such steady solutions are called stationary fronts.

In this paper, we will first establish some qualitative properties of transition fronts connecting 0 and 1 for problem (1.1) and some uniform bounds on their rate of propagation. With the a priori existence of a pulsating front of the type (1.6), we will show that any transition front is equal to this pulsating front up to shift in time. We will finally prove the existence of new types of transition fronts, which are not pulsating fronts satisfying (1.5). In particular, that will show the broadness of the notion of transition fronts and the necessity to introduce and use it even in periodic media.

Some known results

Before going further on, let us recall some known existence results of transition fronts for problem (1.1). We first mention the important case where the function a is equal to a positive constant d, and the function f does not depend on x. In this case, equation (1.1) can be reduced to the homogeneous equation

$$u_t = d u_{xx} + f(u). \tag{1.7}$$

Under the additional assumption that the function $f : [0,1] \to \mathbb{R}$ is such that

$$\begin{cases} f(0) = f(\theta) = f(1) = 0 \text{ for some } \theta \in (0,1), \\ f > 0 \text{ on } (0,\theta), \quad f > 0 \text{ on } (\theta, 1), \quad f'(0) < 0, \quad f'(1) < 0, \end{cases} \tag{1.8}$$

it is well known [4, 19] that there exists a unique speed $c \in \mathbb{R}$ and a unique (up to shifts in x) front

$$u(t,x) = \phi(x - ct) \text{ such that } 0 < \phi < 1 \text{ in } \mathbb{R}, \quad \phi(-\infty) = 1 \text{ and } \phi(+\infty) = 0.$$

Moreover, ϕ is decreasing in x and the speed c has the same sign as the integral $\int_0^1 f$. Such a standard traveling front can be viewed as a pulsating front with average speed c, and a transition front with global mean speed $|c|$ as well. Furthermore, it has been shown recently in [20] that transition fronts connecting 0 and 1 for problem (1.7) are nothing else but the standard traveling fronts $\phi(x - ct)$. We point out that this uniqueness result for the one-dimensional equation (1.7) is not valid for multi-dimensional equations. As a matter of fact, for bistable homogeneous equations in $\mathbb{R}^N$ with $N \geq 2$, the class of transition fronts is larger than that of usual planar traveling fronts: there exist non-planar traveling fronts which are invariant in a moving frame with constant speed [22, 23, 38, 39, 50, 51], as well as some transition fronts which are not invariant in any moving frame as time runs [20]. We also mention [11] for the existence of bistable transition fronts for multi-dimensional homogeneous equations in exterior domains.

For time heterogeneous and space homogeneous bistable equations, the existence and qualitative properties of standard traveling, pulsating or transition fronts have been investigated by Alikakos, Bates and Chen [2], Fang and Zhao [18], and Shen [44, 45, 46, 47]. But, for space heterogeneous bistable equations, there is not so much work on transition fronts. In periodic media, the existence of pulsating fronts for (1.1), under the assumption (1.2), have just been discussed in our previous paper [16]. More precisely, under various additional assumptions on the reaction terms $f(x,u)$ and by using different types of arguments, we proved several existence results of pulsating fronts with nonzero or zero speeds when the spatial period L is small or large (we will come back to the precise assumptions in the comments following Theorem 1.5 below). We also established some properties of the set of periods for which there exist pulsating fronts with nonzero speeds. However, in a given periodic medium, finding a general necessary and sufficient condition for the existence of pulsating fronts with nonzero speeds is still unclear in general. We point out that the existence result is known to hold for all $L > 0$ in some particular cases where $f = f(u)$ does not depend on x and a_L is close to a constant in some norms, see [18, 54, 55, 56], or under various more abstract conditions, see [17, 18]. Apart from the existence results of pulsating fronts, no other existence results of transition fronts have been known for the spatially periodic bistable equation (1.1).

Therefore, it is of particular interest to investigate whether transition fronts of (1.1) are all identically equal to pulsating fronts, or whether there exist transition fronts which are not pulsating fronts. Under the bistability assumption (1.2), the change of sign of $f(x,\cdot)$ in $(0,1)$ and the fact that the roots of $f(x,\cdot)$ in $(0,1)$ depend on x in general may yield the existence of multiple ordered steady states, and thus makes these questions very difficult.

Similar problems have been addressed recently for other types of nonlinearities f, such as ignition type, and Fisher-KPP nonlinearities. Consider for instance an ignition nonlinearity of the type $f(x,u) = g(x)f_0(u)$ for which g is continuous, 1-periodic and positive in $\mathbb{R}$, and $f_0 \geq 0$ in $[0,1]$, $f_0 = 0$ in $[0,\theta] \cup \{1\}$ for some $\theta \in (0,1)$ and f_0 is nonincreasing in $[1-\delta,1]$ for some $\delta > 0$. It is known that for each $L > 0$, (1.1) has a unique (up to shift in t) transition front connecting 0 and 1 in the sense of Definition 1.1, see [32, 33, 40]. Furthermore, this front is actually identically equal to a pulsating front (up to shift in t) with a positive speed, since pulsating fronts are known to exist (and to be unique), see [8]. Similar results have been obtained for multidimensional ignition-type equations, see [60]. On the other hand, for a Fisher-KPP nonlinearity f with $f(x,0) = f(x,1) = 0$ and $0 < f(x,u) \leq \partial_u f(x,0)u$ for $(x,u) \in \mathbb{R} \times (0,1)$, and for each $L > 0$, there exists a positive minimal speed c^*_L in the following sense: (1.1) admits a pulsating front with a speed c if $c \geq c^*_L$, while it does not admit any transition front with a global mean speed c if $c < c^*_L$, see [8, 37, 43, 48, 53]. However, there exist other types of transition fronts which are not standard pulsating fronts (even in the homogeneous media, Fisher-KPP equations with concaves reaction f admit transition fronts which are not of the type $\phi(x-ct)$, see [24, 25] and additional comments after Theorem 1.4 below).

General properties of transition fronts

In this subsection, we present some qualitative properties shared by the solutions of (1.1) in the class of transition fronts, thus showing the robustness of the notion of transition fronts.

Proposition 1.3. *Let u be a transition front connecting 0 and 1 for problem (1.1). Then*

$$\forall\, C \geq 0, \quad 0 < \inf_{t\in\mathbb{R},\, |x|\leq C} u(t, x+\xi_t) \leq \sup_{t\in\mathbb{R},\, |x|\leq C} u(t, x+\xi_t) < 1. \tag{1.9}$$

Furthermore, there is $k_0 \in \mathbb{N}$ such that u is kL-decreasing in x for all integers $k \geq k_0$, that is, $u(t,x) > u(t, x+kL)$ for all $t \in \mathbb{R}$, $x \in \mathbb{R}$ and $k \geq k_0$.

The first property in Proposition 1.3, namely (1.9), shows that in the neighborhood of the positions ξ_t, any transition front is uniformly bounded away from 0 and 1. We refer to [25, 26] for the same type of result for homogeneous or time-dependent monostable equations. Indeed, (1.2) is not needed in the proof of (1.9). Property (1.9) is another way of seeing immediately that the positions $(\xi_t)_{t\in\mathbb{R}}$ of a given transition front u are unique up to an additive bounded function: if $(\widetilde{\xi}_t)_{t\in\mathbb{R}}$ is another family associated with u in the sense of (1.4), then it follows from (1.4) applied to $(\xi_t)_{t\in\mathbb{R}}$ and from (1.9) applied to $(\widetilde{\xi}_t)_{t\in\mathbb{R}}$ that $\sup_{t\in\mathbb{R}} |\xi_t - \widetilde{\xi}_t| < +\infty$.

The second property reveals the effect of the periodicity of the coefficients a_L and f_L on the solutions of (1.1) in the class of transition fronts. In particular, if the transition front u is a pulsating front of type (1.6), then it is L-decreasing in x (that is, one can take $k_0 = 1$) since $c_L \times u$ is increasing in t (see [10, 16]). In addition, all transition fronts are then equal to this pulsating front up to shift in time (see Theorem 1.5 below), and thus they are all L-decreasing in x. Nevertheless, whether all transition fronts for equation (1.1) be L-decreasing in x is not clear in general.

Next, we establish a uniform bound for the propagation rates of all transition fronts of (1.1).

Theorem 1.4. *There is a constant C depending only on the functions f and a such that for any $L > 0$ and any transition front u connecting 0 and 1 for equation (1.1) and associated with $(\xi_t)_{t\in\mathbb{R}}$, there holds*

$$\limsup_{|t-s|\to+\infty} \frac{|\xi_t - \xi_s|}{|t-s|} \leq C. \tag{1.10}$$

Several comments are in order on Theorem 1.4. Firstly, we point out that the bound C in Theorem 1.4 is uniform with respect to all transition fronts for equation (1.1). This uniform boundedness is in sharp contrast with the case of Fisher-KPP nonlinearities, for which there is some minimal speed $c_L^* > 0$ such that for any $c \geq c_L^*$, equation (1.1) has a pulsating front with average speed c, see [8, 53].

We also note that the bound C in Theorem 1.4 is independent of the spatial period L. In particular, if equation (1.1) for a given $L > 0$ has a transition front with a global mean speed γ_L, then γ_L has a uniform bound independent of L. This global boundedness is more general than the local boundedness of the speeds of pulsating fronts we have established in [16, Lemma 3.4].

Finally, we remark that although Theorem 1.4 provides a uniform bound for the global mean speeds of transition fronts, the question of the existence of a global mean speed for a given transition front is still open. However, there is no example showing that equation (1.1), under the assumption (1.2), has a transition front without a global mean speed. On the other hand, in the case of Fisher-KPP nonlinearities, there are examples of transition fronts which do not admit any global mean speed. Consider for instance the homogeneous equation $u_t = u_{xx} + g(u)$ where $g : [0,1] \to \mathbb{R}$ is of class C^2 and concave, $g(0) = g(1) = 0$ and $g(u) > 0$ for all $u \in (0,1)$. This equation admits some transition fronts with positions $(\xi_t)_{t\in\mathbb{R}}$ satisfying $\xi_t/t \to c_-$ as $t \to -\infty$ and $\xi_t/t \to c_+$ as $t \to +\infty$, where c_- and c_+ are any two given real numbers such that $2\sqrt{g'(0)} \le c_- < c_+ < +\infty$, see [24, 25] (see also [26, 59] for further results in time or space dependent media).

Sufficient conditions for the uniqueness of transition fronts

For the homogeneous equation (1.7) with (1.8), it was proved in [20] that the transition fronts are unique up to shifts and equal to the standard traveling front $\phi(x - ct)$, whether c be zero or not. For the periodic equation (1.1), we showed in [16, Theorem 1.1] that for any given period $L > 0$, the speed of pulsating fronts for (1.1) is unique and that, if the speed is not zero, pulsating fronts are unique up to shift in time. For (1.1), it is therefore natural to wonder whether transition fronts can be reduced to stationary or non-stationary pulsating fronts. One of the striking results of this paper is to show that the answer to this question can be yes (see Theorem 1.5 below) or no (see Theorem 1.7).

In this subsection, we provide some sufficient conditions for the uniqueness of solutions in the (a priori larger) class of transition fronts. These conditions also lead to the existence and uniqueness of the global mean speeds. We first give a uniqueness result of Liouville-type under the a priori existence of a pulsating front with nonzero speed.

Theorem 1.5. *If* (1.1) *admits a pulsating front with speed* $c_L \neq 0$*, then any transition front connecting* 0 *and* 1 *is equal to this pulsating front up to shift in time. In particular, any transition front has a global mean speed, equal to* $|c_L|$*.*

As a consequence of Theorem 1.5, we get the uniqueness of transition fronts of equation (1.1) when the spatial period L is small or large, under various assumptions on f. More precisely, if, in addition to (1.2), one assumes that for every $x \in \mathbb{R}$, there exists $\theta_x \in (0,1)$ such that

$$f(x, \theta_x) = 0, \quad f(x,\cdot) < 0 \text{ on } (0, \theta_x) \text{ and } f(x,\cdot) > 0 \text{ on } (\theta_x, 1), \tag{1.11}$$

and that there exists $\overline{\theta} \in (0,1)$ such that

$$\overline{f} < 0 \text{ on } (0, \overline{\theta}),\ \overline{f} > 0 \text{ on } (\overline{\theta}, 1),\ \overline{f}'(\overline{\theta}) > 0, \text{ and } \int_0^1 \overline{f}(u)\,du \neq 0, \tag{1.12}$$

where $\overline{f}(u) = \int_0^1 f(x,u)\,dx$ for $u \in [0,1]$, then from [16, Theorems 1.2, 1.4], there is $L_* > 0$ such that for every $0 < L < L_*$, equation (1.1) admits a pulsating front with nonzero speed, whence any transition front is equal to this pulsating front up to shift in time. In a similar way, it follows from [16, Theorem 1.5] that if, in addition to (1.2) and (1.11), the function f is such that $\int_0^1 f(x,u)\,du > 0$ (resp. < 0) and $\partial f/\partial u(x, \theta_x) > 0$ for all $x \in \mathbb{R}$, then there is $L^* > 0$ such that, for every $L > L^*$, any transition front of equation (1.1) is equal to a pulsating front with positive (resp. negative) speed, and is unique up to shift in time.

In the following theorem, we give a characterization of the pulsating fronts in the class of transition fronts, without assuming a priori the existence of a pulsating front with nonzero speed.

Theorem 1.6. *If there exist a transition front u connecting 0 and 1 and a sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that*

$$\lim_{|t-s|\to+\infty}\Big(\liminf_{n\to+\infty}|\xi_{t+t_n}-\xi_{s+t_n}|\Big)=+\infty, \tag{1.13}$$

then there exists a pulsating front with speed $c_L\neq 0$, and any transition front is then equal to this pulsating front up to shift in time.

Clearly, Theorem 1.6 is more general than Theorem 1.5. Under their assumptions, Theorems 1.5 and 1.6 both imply in particular the existence and uniqueness of the global mean speed in the class of transition front. As a matter of fact, if a transition front u admits a positive global mean speed $c>0$ (as does a pulsating front with a nonzero speed), then

$$\lim_{|t-s|\to+\infty}|\xi_t-\xi_s|=+\infty. \tag{1.14}$$

Theorem 1.6 says that the existence of a transition front u satisfying the rough condition (1.14) (or even a weaker one obtained by passing to the limit along a sequence of time) is actually sufficient for the existence of a pulsating front with nonzero speed. Under this condition, any transition front $\widetilde{u}$ associated with the positions $(\widetilde{\xi}_t)_{t\in\mathbb{R}}$ is then equal to this pulsating front up to shift in time and then satisfies $|\widetilde{\xi}_t-\widetilde{\xi}_s|\sim c|t-s|$ as $|t-s|\to+\infty$ for some $c>0$. In particular, any transition front u which admits a global mean speed $c\geq 0$ is either a pulsating front with nonzero speed or a transition front with null global mean speed and such that $\liminf_{|t-s|\to+\infty}|\xi_t-\xi_s|<+\infty$. We point out that transition fronts with null global mean speed may not be stationary fronts in general, and they may be not unique up to shift in time (see Theorem 1.7 below).

Transition fronts which are not pulsating fronts

In periodic media, pulsating fronts are the natural extension of the standard traveling fronts $\phi(x-ct)$ in homogeneous media. But the notion of pulsating fronts is not broad enough to describe all the transition fronts connecting the steady states 0 and 1. In this subsection, we show indeed that the notion of transition fronts is truly needed, even in periodic media.

To do so, we investigate wave-blocking phenomena for the bistable equation (1.1). Wave-blocking refers to the fact that stationary fronts block the propagation. Some results on wave-blocking phenomena have been obtained in the specific case where the diffusions a_L are not too close to their average and the reactions f_L are x-independent with $f_L(x,u)=f(u)=u(1-u)(u-\theta)$ for some $\theta\simeq 1/2$, see [55, 57] (see also [21] for some blocking phenomena in the case where $f(x,u)$ is compared to a homogeneous bistable function g with nonzero average). Moreover, wave-blocking phenomena have been extensively investigated for various other bistable models, see, e.g., [1, 6, 12, 14, 28, 31] for spatially discrete models, [3, 30, 36, 41] for some non-periodic equations, and [7, 13] for some higher-dimensional equations. In the present paper, for constant diffusions a_L and some x-dependent reactions f_L, we prove that wave-blocking occurs when the period L is large, and furthermore we obtain the existence of transition fronts connecting 0 and 1 between two ordered stationary fronts. We state the main result as follows.

Theorem 1.7. *Assume, in addition to* (1.2)*, that f satisfies* (1.11)*, the functions a, $\partial_u f(\cdot,0)$ and $\partial_u f(\cdot,1)$ are constants, and*

$$\min_{x\in\mathbb{R}}\Big(\int_0^1 f(x,u)du\Big)<0 \quad \textit{and} \quad \max_{x\in\mathbb{R}}\Big(\int_0^1 f(x,u)du\Big)>0. \tag{1.15}$$

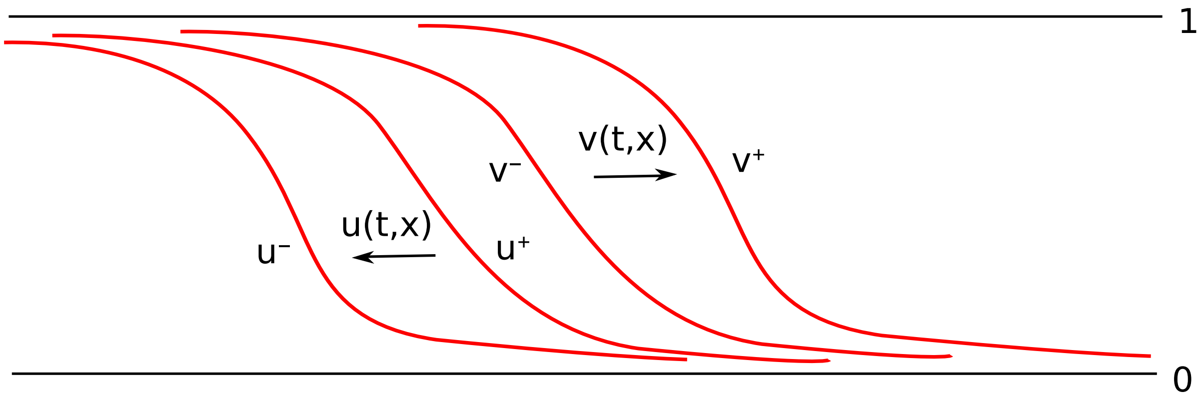

Figure 1: Stationary fronts $0 < u^-(x) < u^+(x) \leq v^-(x) < v^+(x) < 1$ and transition fronts u and v such that $u^-(x) < u(t,x) < u^+(x)$ and $v^-(x) < v(t,x) < v^+(x)$.

Then there is $L^* > 0$ such that for any $L > L^*$, equation (1.1) *with the period L admits both some stationary fronts $u^\pm$ and $v^\pm$ satisfying $0 < u^-(x) < u^+(x) \leq v^-(x) < v^+(x) < 1$ for all $x \in \mathbb{R}$, as well as some transition fronts u and v connecting 0 and 1 and such that*

$$\begin{cases} u(-\infty,x) = u^+(x), & u(+\infty,x) = u^-(x), \\ v(-\infty,x) = v^-(x), & v(-\infty,x) = v^+(x), \end{cases} \quad \textit{uniformly in } x \in \mathbb{R}.$$

Furthermore, $u(t,x)$ is decreasing in t, while $v(t,x)$ is increasing in t. Lastly, for any transition front $\widetilde{u}$ associated with positions $(\xi_t)_{t\in\mathbb{R}}$, there is $M \geq 0$ such that $|\xi_t - \xi_s| \leq M$ for all $(t,s) \in \mathbb{R}^2$. In particular, all transition fronts have zero global mean speed and there are transition fronts which are not pulsating fronts.

Let us first explain the phenomenon behind Theorem 1.7. On the one hand, when the period L is large, thanks to (1.15), there are large intervals in, say, the positive direction (in x) where the nonlinearity $f_L(x,\cdot)$ is locally close to a homogeneous bistable nonlinearity with a negative integral over $[0,1]$. Roughly speaking, this means that around such large intervals, the speed of any solution of the associated Cauchy problem with a front-like initial condition is (strictly) negative, at least on some interval of time. In other words, any solution is blocked by a supersolution moving leftwards and located far on the right. Similarly, any solution is blocked by a subsolution moving rightwards and located far on the left. The supersolution and the subsolution are ordered and move towards each other as time runs. Therefore, some ordered stationary fronts exist. On the other hand, the supersolution (resp. subsolution) only blocks the convergence to 1 (resp. 0) as $t \to +\infty$ of the solution, but not the convergence to some stationary fronts. Thus, there exist time-monotone solutions which connect two ordered stationary fronts as $t \to \pm\infty$ (see Figure 1). Such solutions converge to the stable steady states 0 and 1 as $x \to \pm\infty$ uniformly in t and therefore they are non-stationary transition fronts with null global mean speed. Furthermore, notice that, since (1.1) is L-periodic in x, Theorem 1.7 actually provides the existence of two sequences of stationary fronts $(u_n^\pm)_{n\in\mathbb{Z}} = (u^\pm(nL+\cdot))_{n\in\mathbb{Z}}$ and $(v_n^\pm)_{n\in\mathbb{Z}} = (v^\pm(nL+\cdot))_{n\in\mathbb{Z}}$ and two sequences of time-monotone non-stationary transitions fronts $(u_n)_{n\in\mathbb{Z}} = (u(\cdot,nL+\cdot))_{n\in\mathbb{Z}}$ and $(v_n)_{n\in\mathbb{Z}} = (v(\cdot,nL+\cdot))_{n\in\mathbb{Z}}$ converging to the stationary fronts $u_n^\pm$ and $v_n^\pm$ as $t \to \pm\infty$.

Theorem 1.7 actually shows the broadness of Definition 1.1, since the class of transition fronts includes time-global solutions which connect 0 and 1 in space and connect two strictly ordered stationary fronts in time. Notice that this type of solutions cannot be described by non-stationary pulsating fronts, since any such pulsating front connects 0 and 1 both in space and in time. It is also worth pointing out that in the homogeneous case where a and f do not depend on x, the transition fronts are noting else but the standard traveling fronts, see [20]. As far as we know, Theorem 1.7 provides the first example of transition fronts in periodic media, which are neither stationary nor non-stationary pulsating fronts. It shows in particular the larger complexity of the dynamics in periodic media than in homogeneous media.

Theorem 1.7 also provides some examples of transition fronts which are not unique up to shift in time. Indeed, under the assumptions of Theorem 1.7, for any $L > L^*$, some transition fronts of (1.1) are increasing in time, others are decreasing and other are stationary. But whether all the transition fronts are monotone in time is still unclear. Moreover, some other natural questions arise: for a given $L > L^*$, does any transition front converge to some stationary fronts as $t \to \pm\infty$? Under the assumptions in Theorem 1.7 and the assumption (1.12), is it true that $L^* = L_*$, if L_* denotes the smallest such real number for which the conclusion of Theorem 1.7 holds and L_* denotes the largest real number for which equation (1.1) admits a pulsating front with nonzero speed ?

Generally speaking, for diffusion and reaction coefficients $a(x)$ and $f(x, u)$ satisfying (1.2), the questions of the existence of front-like solutions connecting 0 and 1 and of the existence of propagation speeds are still open, even under the additional assumption (1.11). Based on the work in this present paper and the companion paper [16], one can conclude that the spatial period L plays an important role in answering these questions. Namely, under various assumptions, there are transition fronts propagating with nonzero speeds (pulsating fronts with nonzero speeds) when L is small, while there are both stationary and non-stationary transition fronts propagating with zero speed when L is large. However, so far there is no explicit condition to guarantee the existence of transition fronts in general, and no example is known to show their non-existence.

Outline of the paper. Section 2 is devoted to the proof of the general properties of transition fronts, that is, Proposition 1.3 and Theorem 1.4. In Section 3, we prove Theorems 1.5 and 1.6 on the uniqueness of transition fronts. Lastly, Section 4 is devoted to the proof of Theorem 1.7, that is, the existence of stationary fronts and non-stationary transition fronts which are not standard pulsating fronts.

2 General properties of transition fronts

This section is devoted to the proofs of Proposition 1.3 and Theorem 1.4 on the kL-spatial monotonicity of the transition fronts and the boundedness of the propagation rates.

2.1 Spatial monotonicity: proof of Proposition 1.3

We first do the proof of Proposition 1.3. The first statement will follow easily from Definition 1.1, that is, u converges to the steady states 0 or 1 far away from the positions ξ_t uniformly in t. The proof of the second statement of Proposition 1.3 is more involved. We will slide u with respect to the x-variable on the set $L\mathbb{Z}$. Before doing so, we first present a result on the uniform boundedness of the local oscillations of $(\xi_t)_{t\in\mathbb{R}}$.

Lemma 2.1. *For any transition front u connecting 0 and 1 for problem (1.1), there holds*

$$\forall\,\tau>0,\qquad \sup_{(t,s)\in\mathbb{R}^2,\,|t-s|\le\tau}|\xi_t-\xi_s|<+\infty. \tag{2.1}$$

Proof. The proof is an immediate application of [25, Proposition 4.1], and we omit the details. □

This property holds for more general heterogeneous one-dimensional equations with other types of nonlinearities, and it has its own interest. As an application of Lemma 2.1, by applying recursively (2.1) with $\tau=1$, one infers that $\limsup_{t\to\pm\infty}|\xi_t/t|<+\infty$. Namely, the propagation speeds of any transition front are asymptotically bounded as $t\to\pm\infty$.

Proof of Proposition 1.3. We only show the first inequality in (1.9), since the last one follows similar lines, and the second one is obvious. Assume, by contradiction, that there exists a sequence $(t_n,x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^2$ such that $(x_n-\xi_{t_n})_{n\in\mathbb{R}}$ is bounded and $u(t_n,x_n)\to0$ as $n\to+\infty$. Write $x_n=x'_n+x''_n$ with $x'_n\in L\mathbb{Z}$ and $x''_n\in[0,L)$, and set

$$u_n(t,x)=u(t+t_n,x+x'_n)\ \text{ for } (t,x)\in\mathbb{R}^2 \text{ and } n\in\mathbb{N}.$$

Since the functions a_L and f_L are independent of t and L-periodic in x, each function u_n obeys equation (1.1). Up to extraction of some subsequence, one can assume that $x''_n\to x_\infty\in[0,L]$ as $n\to+\infty$ and that, from standard parabolic estimates, $u_n(t,x)\to u_\infty(t,x)$ as $n\to+\infty$ locally uniformly in $\mathbb{R}^2$, where $0\le u_\infty\le1$ solves (1.1). Furthermore, since $u(t_n,x_n)\to0$ as $n\to+\infty$, one has $u_\infty(0,x_\infty)=\lim_{n\to+\infty}u_n(0,x''_n)=\lim_{n\to+\infty}u(t_n,x_n)=0$. It then follows from the strong maximum principle that $u_\infty\equiv0$. On the other hand, from Definition 1.1, there exists $M>0$ such that $u(t,x)\ge1/2$ for all $x-\xi_t\le-M$ and $t\in\mathbb{R}$. One then infers that

$$u_n(0,x)\ge\frac12\ \text{ for all } x+x_n-x''_n-\xi_{t_n}\le-M \text{ and } t\in\mathbb{R}.$$

Remember that the sequences $(x_n-\xi_{t_n})_{n\in\mathbb{R}}$ and $(x''_n)_{n\in\mathbb{N}}$ are bounded. Thus, there exists $M_1\in\mathbb{R}$ such that $u_\infty(0,M_1)\ge1/2$, which is impossible. Hence, the proof of the first inequality in (1.9) is complete.

Let us now turn to the proof of the second assertion of Proposition 1.3. From Definition 1.1, there is $B>0$ such that

$$\forall\,(t,x)\in\mathbb{R}^2,\qquad \begin{cases} x-\xi_t\ge B & \Longrightarrow\ 0<u(t,x)\le\delta,\\ x-\xi_t\le-B & \Longrightarrow\ 1-\delta\le u(t,x)<1,\end{cases} \tag{2.2}$$

where $\delta\in(0,1/2)$ is the constant in (1.2). Since ∂_uf is continuous in $\mathbb{R}\times[0,1]$ and periodic in x (∂_uf is thus uniformly continuous in $\mathbb{R}\times[0,1]$), one can also assume without loss of generality, even if it means decreasing $\delta>0$, that, for every $x\in\mathbb{R}$, $f(x,\cdot)$ is decreasing in $[0,\delta]$ and $[1-\delta,1]$, and then in $(-\infty,\delta]$ and $[1-\delta,+\infty)$ owing to the definition of the extension of f on $\mathbb{R}\times(\mathbb{R}\backslash[0,1])$. Choose $k_0\in\mathbb{N}$ large enough such that $k_0L\ge2B$ and fix any $k\in\mathbb{N}$ with $k\ge k_0$. One then sees from the second inequality of (2.2) that

$$v(t,x):=u(t,x-kL)\ge1-\delta\ge\delta\ \text{ for all } x-\xi_t\le B. \tag{2.3}$$

This together with the first inequality of (2.2) yields

$$v(t,x)\ge u(t,x)\ \text{ when } x-\xi_t=B. \tag{2.4}$$

Since the function u is bounded, the nonnegative real number

$$\varepsilon^* := \inf\left\{\varepsilon > 0\,;v(t,x) \geq u(t,x) - \varepsilon \text{ for all } (t,x) \in \mathbb{R}^2\right\}$$

is well-defined. It is obvious that

$$u(t, x - kL) = v(t,x) \geq u(t,x) - \varepsilon^* \text{ for all } (t,x) \in \mathbb{R}^2. \tag{2.5}$$

In order to prove that $u(t,x)$ is kL-decreasing in x, it is sufficient to show $\varepsilon^* = 0$. Indeed, if $\varepsilon^* = 0$, then $u(t,x-kL) \geq u(t,x)$ for all $(t,x) \in \mathbb{R}^2$ and since both functons $u(t,x-kL)$ and $u(t,x)$ satisfy the same equation (1.1) (because a_L and f_L are L-periodic in x), the strong maximum principle implies that either $u(t,x-kL) > u(t,x)$ for all $(t,x) \in \mathbb{R}^2$, or $u(t,x-kL) = u(t,x)$ for all $(t,x) \in \mathbb{R}^2$. The latter is impossible since $k \geq k_0 > 0$ and $u(t,-\infty) = 1 > 0 = u(t,+\infty)$ for all $t \in \mathbb{R}$.

Assume, by contradiction, that $\varepsilon^* > 0$. Then there exist a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive real numbers and a sequence (t_n, x_n) in $\mathbb{R}^2$ such that

$$\varepsilon_n \to \varepsilon^* \text{ as } n \to +\infty \quad \text{and} \quad v(t_n,x_n) < u(t_n,x_n) - \varepsilon_n \text{ for all } n \in \mathbb{N}. \tag{2.6}$$

We first claim that the sequence $(x_n - \xi_{t_n})_{n\in\mathbb{N}}$ is bounded. Otherwise, up to extraction of some subsequence, it would converge to $+\infty$ or $-\infty$ as $n \to +\infty$. If $x_n - \xi_{t_n} \to +\infty$ as $n \to +\infty$, then, from Definition 1.1, one has $u(t_n,x_n) \to 0$ and $v(t_n,x_n) = u(t_n,x_n - kL) \to 0$ as $n \to +\infty$. Thus, $v(t_n,x_n) - u(t_n,x_n) \to 0$ as $n \to +\infty$, which contradicts (2.6) and the positivity of ε^*. Similarly, the case $\lim_{n\to+\infty}(x_n - \xi_{t_n}) = -\infty$ leads to a contradiction. Therefore, the sequence $(x_n - \xi_{t_n})_{n\in\mathbb{N}}$ is bounded.

Next, we set

$$w(t,x) := v(t,x) - (u(t,x) - \varepsilon^*) = u(t,x-kL) - u(t,x) + \varepsilon^* \text{ for } (t,x) \in \mathbb{R}^2.$$

Clearly, $w \geq 0$ in $\mathbb{R}^2$ from (2.5). Define now $E_1 := \left\{(t,x) \in \mathbb{R}^2\,;\ x - \xi_t \geq B\right\}$ and $E_2 := \left\{(t,x) \in \mathbb{R}^2\,;\ x - \xi_t \leq B\right\}$. As a consequence of (2.2) and (2.3), one has $u(t,x) - \varepsilon^* < u(t,x) \leq \delta$ for all $(t,x) \in E_1$, and $v(t,x) + \varepsilon^* > v(t,x) \geq 1 - \delta$ for all $(t,x) \in E_2$. Since $f_L(x,\cdot)$ is decreasing in $(-\infty, \delta]$, one infers that

$$\begin{aligned}(u - \varepsilon^*)_t &= (a_L(x)(u-\varepsilon^*)_x)_x + f_L(x,u)\\ &\leq (a_L(x)(u-\varepsilon^*)_x)_x + f_L(x, u - \varepsilon^*) \text{ for all } (t,x) \in E_1.\end{aligned}$$

Namely, the function $u - \varepsilon^*$ is a subsolution of (1.1) in the set E_1. Since v solves (1.1), it follows that

$$w_t \geq (a_L(x)w_x)_x + \frac{f_L(x,v) - f_L(x,u-\varepsilon^*)}{v - (u - \varepsilon^*)} w \text{ in } E_1,$$

where the quotient is defined as $\partial_u f_L(x, v(t,x))$ for any point $(t,x) \in E_1$ such that $v(t,x) = u(t,x) - \varepsilon^*$. Similarly, the function $v + \varepsilon^*$ is a supersolution of (1.1) in the set E_2, and then w satisfies

$$w_t \geq (a_L(x)w_x)_x + \frac{f_L(x,v+\varepsilon^*) - f_L(x,u)}{(v+\varepsilon^*) - u} w \text{ in } E_2.$$

Finally, since $\partial_u f(x,u)$ is bounded in $\mathbb{R}^2$, the function w satisfies an inequation of the type

$$w_t \geq (a_L(x)w_x)_x + b(t,x)w \text{ for all } (t,x) \in \mathbb{R}^2$$

for some bounded function $b:\mathbb{R}^2\to\mathbb{R}$. Furthermore, from standard parabolic estimates, the function $w=v-u+\varepsilon^*$ has bounded first-order derivatives. Since the sequence $(x_n-\xi_{t_n})_{n\in\mathbb{N}}$ is bounded, it follows from Lemma 2.1 that, for any $\tau>0$, the sequence $(x_n-\xi_{t_n-\tau}-B)_{n\in\mathbb{N}}$ is bounded. Since w is nonnegative in $\mathbb{R}^2$ and $w(t_n,x_n)\to 0$ as $n\to+\infty$ from (2.5) and (2.6), one then concludes from Krylov-Safonov-Harnack type inequalities that, for any $\tau>0$, $w(t_n-\tau,\xi_{t_n-\tau}+B)\to 0$ as $n\to+\infty$, which contradicts (2.4) and the positivity of ε^*. Hence, $\varepsilon^*=0$ and the proof of Proposition 1.3 is thereby complete. □

2.2 Global boundedness of the propagation rates: proof of Theorem 1.4

This section is devoted to the proof of Theorem 1.4 on global bounds for the propagation rates of any transition front connecting 0 and 1 for (1.1). The general idea can be summarized as follows. From the first part of Proposition 1.3, any transition front u ranges in $(\varepsilon_0,1-\varepsilon_0)$ on the set $\big\{(t,\xi_t);\;\; t\in\mathbb{R}\big\}$, for some $\varepsilon_0\in(0,1/2)$. We will show that there exists a constant $C>0$, independent of u and of L, such that for any fixed $t_0\in\mathbb{R}$, $u(t,x)$ is less than ε_0 when $t-t_0\gg 1$ and $x-\xi_{t_0}\gg C(t-t_0)$, and is larger than $1-\varepsilon_0$ when $t-t_0\gg 1$ and $x-\xi_{t_0}\ll -C(t-t_0)$. This guarantees that, up to some bounded shifts, the positions ξ_t stay in the expanding interval $[\xi_{t_0}-C(t-t_0),\,\xi_{t_0}+C(t-t_0)]$ as $t-t_0$ is large. In other words, C gives an upper bound for the propagation speed of the positions ξ_t, while $-C$ provides a lower bound. To get the constant C, we will construct, for every $L>0$, a supersolution (resp. subsolution) for (1.1), which approaches the limiting state 0 (resp. 1) exponentially fast as $t\to+\infty$ and $x\to+\infty$ (resp. $x\to-\infty$), the exponential rates being independent of $L>0$.

Before doing so, let us first recall some properties of the principal eigenvalue of some linear second-order differential operators. For every $L>0$ and $\mu\in\mathbb{R}$, let $\mathcal{T}_{L,\mu}$ be the linear operator defined on $C^2_L:=\big\{\psi\in C^2(\mathbb{R})\;;\;\psi(x+L)=\psi(x)\text{ for all }x\in\mathbb{R}\big\}$ by

$$\mathcal{T}_{L,\mu}[\psi]=(a_L\psi')'+2\mu a_L\psi'+(\mu^2a_L+\mu a_L')\psi.$$

The Krein-Rutman theory provides the existence and uniqueness of the principal eigenvalue $\lambda(L,\mu)$ of $\mathcal{T}_{L,\mu}$, associated with a (unique up to multiplication) positive eigenfunction $\psi_{L,\mu}\in C^2_L$. One can normalize $\psi_{L,\mu}$ in such a way that $\|\psi_{L,\mu}\|=1$ with the norm $\|\cdot\|:=\|\cdot\|_{L^\infty(\mathbb{R})}$. Furthermore, for any fixed $\mu\in\mathbb{R}$, it is known from [35] that $\lambda(L,\mu)$ is nondecreasing in $L>0$. On the other hand, integrating both sides of $\mathcal{T}_{L,\mu}[\psi_{L,\mu}]=\lambda(L,\mu)\psi_{L,\mu}$ over $[0,L]$ yields $\lambda(L,\mu)\le\|a\|\mu^2+\|a'\||\mu|/L$ for all $L>0$ and $\mu\in\mathbb{R}$. As a consequence,

$$\lambda(L,\mu)\le\lim_{L'\to+\infty}\lambda(L',\mu)\le\|a\|\mu^2\;\text{ for all }L>0\text{ and }\mu\in\mathbb{R}.\tag{2.7}$$

Next we construct a supersolution for equation (1.1).

Lemma 2.2. *There is a constant $C>0$, depending only on a and f, such that for every $L>0$, the function $\overline{u}$ defined by*

$$\overline{u}(t,x)=\min\Big(\mathrm{e}^{-(x-Ct)}\,\psi_{L,-1}(x)+\frac{\delta}{2}\,\mathrm{e}^{-\gamma t},1\Big)\;\text{ for }t\ge 0\text{ and }x\in\mathbb{R},\tag{2.8}$$

is a supersolution of equation (1.1), where $\psi_{L,-1}$ is the principal eigenfunction of $\mathcal{T}_{L,-1}$, and $\delta\in(0,1/2)$ and $\gamma>0$ are given in (1.2).

Proof. Set $K = \max_{(x,u)\in\mathbb{R}\times[0,1]} |\partial_u f(x,u)|$ and let $C > 0$ be given by

$$C = \|a\| + \gamma + 2K \tag{2.9}$$

(remember that $\max(\partial_u f(x,0), \partial_u(x,1)) \le -\gamma < 0$ for all $x \in \mathbb{R}$, whence $\gamma \le K$ and the computations below would also work with $C = \|a\| + 3K$).

Now, let us check that

$$\overline{N}(t,x) := \overline{u}_t(t,x) - (a_L(x)\overline{u}_x(t,x))_x - f_L(x, \overline{u}(t,x)) \ge 0$$

for all $t \ge 0$ and $x \in \mathbb{R}$ such that $\overline{u}(t,x) < 1$. This will be sufficient to ensure that $\overline{u}$ is a supersolution, since $f_L(\cdot, 1) = 0$. From (2.7) and the definition of $\lambda(L,-1)$, it is straightforward to check that, when $\overline{u}(t,x) < 1$,

$$\begin{aligned}
\overline{N}(t,x) &= -\big((a_L\psi'_{L,-1})'(x) - 2a_L(x)\psi'_{L,-1}(x) + (a_L(x) - a'_L(x))\psi_{L,-1}(x)\big)\,\mathrm{e}^{-(x-Ct)} \\
&\quad + C\,\mathrm{e}^{-(x-Ct)}\,\psi_{L,-1}(x) - \frac{\delta\gamma}{2}\mathrm{e}^{-\gamma t} - f_L(x, \overline{u}(t,x)) \\
&= -\big(\lambda(L,-1) - C\big)\,\mathrm{e}^{-(x-Ct)}\,\psi_{L,-1}(x) - \frac{\delta\gamma}{2}\mathrm{e}^{-\gamma t} - f_L(x, \overline{u}(t,x)) \\
&\ge -\big(\|a\| - C\big)\,\mathrm{e}^{-(x-Ct)}\,\psi_{L,-1}(x) - \frac{\delta\gamma}{2}\mathrm{e}^{-\gamma t} - f_L(x, \overline{u}(t,x)).
\end{aligned} \tag{2.10}$$

If $0 < \overline{u}(t,x) \le \delta$, then $f_L(x, \overline{u}(t,x)) \le -\gamma\overline{u}(t,x)$ from (1.2), whence

$$\overline{N}(t,x) \ge -\big(\|a\| - C - \gamma\big)\,\mathrm{e}^{-(x-Ct)}\,\psi_{L,-1}(x) \ge 0$$

from (2.9) and the positivity of γ and K. On the other hand, if $\delta \le \overline{u}(t,x) < 1$, then $f_L(x, \overline{u}(t,x)) = f_L(x, \overline{u}(t,x)) - f_L(x,0) \le K\overline{u}(t,x)$. It then follows that

$$\delta \le \overline{u}(t,x) = \mathrm{e}^{-(x-Ct)}\,\psi_{L,-1}(x) + \frac{\delta}{2}\mathrm{e}^{-\gamma t} \le \mathrm{e}^{-(x-Ct)}\,\psi_{L,-1}(x) + \frac{\delta}{2},$$

whence $(\delta/2)\,\mathrm{e}^{-\gamma t} \le \delta/2 \le \mathrm{e}^{-(x-Ct)}\,\psi_{L,-1}(x)$. Thus, if $\delta \le \overline{u}(t,x) < 1$, then (2.10) yields

$$\overline{N}(t,x) \ge -\big(\|a\| - C + K\big)\,\mathrm{e}^{-(x-Ct)}\,\psi_{L,-1}(x) - \frac{\delta}{2}\,(\gamma + K)\,\mathrm{e}^{-\gamma t},$$

whence $\overline{N}(t,x) \ge -\big(\|a\| - C + \gamma + 2K\big)\,\mathrm{e}^{-(x-Ct)}\,\psi_{L,-1}(x) = 0$ from (2.9). Hence, $\overline{u}$ is a supersolution of (1.1). The proof of Lemma 2.2 is thereby complete. □

The supersolution $\overline{u}$ depends on period L, but its exponential decay rates as $t \to +\infty$ and $x \to +\infty$ are independent of L. In a similar way, we can construct a subsolution.

Lemma 2.3. *For every $L > 0$, the function $\underline{u}$ defined by*

$$\underline{u}(t,x) = \max\Big(1 - \mathrm{e}^{x+Ct}\,\psi_{L,1}(x) - \frac{\delta}{2}\,\mathrm{e}^{-\gamma t}, 0\Big) \quad \text{for } t \ge 0 \text{ and } x \in \mathbb{R}, \tag{2.11}$$

is a subsolution of equation (1.1), where $\psi_{L,1}$ is the principal eigenfunction of $\mathcal{T}_{L,1}$, $\delta \in (0, 1/2)$ and $\gamma > 0$ are given in (1.2), and $C > 0$ is given in (2.9).

Proof. The proof of this lemma is similar to that of Lemma 2.2. For the sake of completeness, we include the details here. To prove that $\underline{u}$ is a subsolution of equation (1.1), one only needs to check that

$$\underline{N}(t,x) := \underline{u}_t(t,x) - (a_L(x)\underline{u}_x(t,x))_x - f_L(x,\underline{u}(t,x)) \le 0$$

for all $t \ge 0$ and $x \in \mathbb{R}$ such that $\underline{u}(t,x) > 0$, since $f_L(\cdot,0) = 0$. From (2.7) and the definition of $\lambda(L,1)$, it is straightforward to check that

$$\begin{aligned}\underline{N}(t,x) &= \big(\lambda(L,1) - C\big)\,\mathrm{e}^{x+Ct}\,\psi_{L,1}(x) + \frac{\delta\gamma}{2}\mathrm{e}^{-\gamma t} - f_L(x,\underline{u}(t,x))\\ &\le \big(\|a\|^2 - C\big)\,\mathrm{e}^{x+Ct}\,\psi_{L,1}(x) + \frac{\delta\gamma}{2}\mathrm{e}^{-\gamma t} - f_L(x,\underline{u}(t,x)).\end{aligned}$$

If $1-\delta \le \underline{u}(t,x) < 1$, then $f_L(x,\underline{u}(t,x)) \ge \gamma(1-\underline{u}(t,x))$ from (1.2), whence

$$\underline{N}(t,x) \le \big(\|a\|^2 - C - \gamma\big)\,\mathrm{e}^{x+Ct}\,\psi_{L,1}(x) \le 0.$$

On the other hand, if $0<\underline{u}(t,x)\le 1-\delta$, then $f_L(x,\underline{u}(t,x)) = f_L(x,\underline{u}(t,x)) - f_L(x,1) \ge -K(1-\underline{u}(t,x))$. In that case, $1-\mathrm{e}^{x+Ct}\,\psi_{L,1}(x) - \delta/2 \le 1-\mathrm{e}^{x+Ct}\,\psi_{L,1}(x) - (\delta/2)\mathrm{e}^{-\gamma t} = \underline{u}(t,x) \le 1-\delta$, whence $(\delta/2)\,\mathrm{e}^{-\gamma t} \le \delta/2 \le \mathrm{e}^{x+Ct}\,\psi_{L,1}(x)$. Thus, if $0 < \underline{u}(t,x) \le 1-\delta$, then

$$\begin{aligned}\underline{N}(t,x) &\le \big(\|a\|^2 - C + K\big)\,\mathrm{e}^{x+Ct}\,\psi_{L,1}(x) + \frac{\delta}{2}\,(\gamma + K)\,\mathrm{e}^{-\gamma t}\\ &\le \big(\|a\|^2 - C + \gamma + 2K\big)\,\mathrm{e}^{x+Ct}\,\psi_{L,1}(x) = 0.\end{aligned}$$

Hence, $\underline{u}$ is a subsolution of (1.1) and the proof of Lemma 2.3 is thereby complete. □

Given the previous two lemmas, we are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. Let $u(t,x)$ be a transition front connecting 0 and 1 for problem (1.1) with period $L > 0$, associated with positions $(\xi_t)_{t\in\mathbb{R}}$ satisfying (1.4). We will prove the inequality (1.10) with the constant C given in (2.9) and used in Lemmas 2.2 and 2.3. To do so, we shall show an upper bound for $\limsup_{|t-s|\to+\infty}(\xi_t-\xi_s)/(t-s)$ and a lower bound for $\liminf_{|t-s|\to+\infty}(\xi_t-\xi_s)/(t-s)$.

First of all, for each $k \in \mathbb{Z}$, there is a unique $n_k \in \mathbb{Z}$ such that $n_k L \le \xi_k < (n_k+1)L$. For all $k \in \mathbb{Z}$ and $t \in [k,k+1)$, by defining $\widetilde{\xi}_t = n_k L$, it follows from (2.1) applied with $\tau = 1$ that $\sup_{t\in\mathbb{R}} |\xi_t - \widetilde{\xi}_t| < +\infty$, whence property (1.4) in Definition 1.1 still holds with the family $(\widetilde{\xi}_t)_{t\in\mathbb{R}}$. Therefore, even if it means redefining ξ_t, one can assume without loss of generality that $\xi_t \in L\mathbb{Z}$ for all $t \in \mathbb{R}$ (and even that ξ_t is constant on every interval $[k,k+1)$ with $k \in \mathbb{Z}$).

Step 1: the upper estimate. Let $\overline{u}$ be the function given in (2.8) with the constant C defined in (2.9). We shall show that

$$\limsup_{|t-s|\to+\infty} \frac{\xi_t - \xi_s}{t-s} \le C. \tag{2.12}$$

From Definition 1.1, there is $C_1 > 0$ such that

$$u(t, x+\xi_t) \le \frac{\delta}{2} \ \text{ for all } x \ge C_1 \text{ and } t \in \mathbb{R}. \tag{2.13}$$

Owing to the definition of $\overline{u}$ in (2.8), there is $x_0 < 0$ such that $\overline{u}(0,x) = 1$ for all $x \le x_0$. Let $M_1 \in L\mathbb{Z}$ be such that $M_1 \ge C_1 - x_0$. Then $0 < u(t,x+\xi_t) < 1 = \overline{u}(0,x-M_1)$ for all $x \le C_1$ and $t \in \mathbb{R}$.

On the other hand, it follows from (2.13) that $0 < u(t, x + \xi_t) \leq \delta/2 < \overline{u}(0, x - M_1)$ for all $x \geq C_1$ and $t \in \mathbb{R}$. Therefore, $u(t, x + \xi_t) < \overline{u}(0, x - M_1)$ for all $x \in \mathbb{R}$ and $t \in \mathbb{R}$. Since equation (1.1) is L-periodic in x and since ξ_t (for every $t \in \mathbb{R}$) and M_1 belong to $L\mathbb{Z}$, it then follows from Lemma 2.2 and the maximum principle that

$$0 < u(s, x + \xi_t) < \overline{u}(s - t, x - M_1) \ \text{ for all } t \leq s \text{ and } x \in \mathbb{R}. \tag{2.14}$$

Now let us assume by contradiction that there are a real number $c_1 \in (C, +\infty)$ and two sequences $(t_k)_{k\in\mathbb{N}}$ and $(s_k)_{k\in\mathbb{N}}$ of real numbers such that $|t_k - s_k| \to +\infty$ as $k \to +\infty$ and

$$\frac{\xi_{t_k} - \xi_{s_k}}{t_k - s_k} > c_1 \ \text{ for all } k \in \mathbb{N}. \tag{2.15}$$

Without loss of generality, one can assume that $t_k < s_k$ for all $k \in \mathbb{N}$. It then follows from (2.14) that $0 < u(s_k, x + \xi_{t_k}) < \overline{u}(s_k - t_k, x - M_1)$ for all $k \in \mathbb{N}$ and $x \in \mathbb{R}$. Notice that $\lim_{k\to+\infty} \overline{u}(s_k - t_k, c_1(s_k - t_k) - M_1) = 0$ by definition of $\overline{u}$ in (2.8), since $c_1 > C$. This yields

$$\lim_{k\to+\infty} u(s_k, c_1(s_k - t_k) + \xi_{t_k}) = 0$$

Set $\varepsilon_1 := \inf_{(s,x)\in\mathbb{R}\times(-\infty,0]} u(s, x + \xi_s)$. Clearly, $\varepsilon_1 > 0$ from Definition 1.1 and Proposition 1.3. Since there is $k_1 \in \mathbb{N}$ such that $u(s_k, c_1(s_k - t_k) + \xi_{t_k}) < \varepsilon_1$ for all $k \geq k_1$, one finally gets that $c_1(s_k - t_k) + \xi_{t_k} \geq \xi_{s_k}$ for all $k \geq k_1$, which is a contradiction with (2.15) (remember that $t_k < s_k$). Hence, the proof of inequality (2.12) is finished.

Step 2: the lower estimate. Let $\underline{u}$ be the function given in (2.11) with the constant C defined in (2.9). Here we show that

$$\liminf_{|t-s|\to+\infty} \frac{\xi_t - \xi_s}{t - s} \geq -C. \tag{2.16}$$

The proof is similar to that of (2.12). Namely, from Definition 1.1, there is $C_2 > 0$ such that $1 > u(t, x + \xi_t) \geq 1 - \delta/2$ for all $x \leq -C_2$ and $t \in \mathbb{R}$. Then owing to the definition of $\underline{u}$, there is $M_2 \in L\mathbb{Z}$ such that $1 > u(t, x + \xi_t) > \underline{u}(0, x + M_2)$ for all $x \in \mathbb{R}$ and $t \in \mathbb{R}$. It follows then from Lemma 2.2 and the maximum principle that

$$1 > u(s, x + \xi_s) > \underline{u}(s - t, x + M_2) \ \text{ for all } t \leq s \text{ and } x \in \mathbb{R}. \tag{2.17}$$

Let us assume by contradiction that there are a positive constant $c_2 > C$ and two sequences $(t'_k)_{k\in\mathbb{N}}$ and $(s'_k)_{k\in\mathbb{N}}$ of real numbers such that $|t'_k - s'_k| \to +\infty$ as $k \to +\infty$ and

$$\frac{\xi_{t'_k} - \xi_{s'_k}}{t'_k - s'_k} < -c_2 \ \text{ for all } k \in \mathbb{N}. \tag{2.18}$$

Without loss of generality, one can assume that $t'_k < s'_k$ for all $k \in \mathbb{N}$. It then follows from (2.17) that $1 > u(s'_k, x + \xi_{t'_k}) > \underline{u}(s'_k - t'_k, x + M_2)$ for all $k \in \mathbb{N}$ and $x \in \mathbb{R}$. Notice that $\lim_{k\to+\infty} \underline{u}(s'_k - t'_k, -c_2(s'_k - t'_k) + M_2) = 1$ by definition of $\underline{u}$ in (2.11), and since $c_2 > C$. Therefore, $\lim_{k\to+\infty} u(s'_k, -c_2(s'_k - t'_k) + \xi_{t'_k}) = 1$. Set $\varepsilon_2 := \inf_{(s,x)\in\mathbb{R}\times[0,+\infty)}(1 - u(s, x + \xi_s)) > 0$. Then there is $k_2 \in \mathbb{N}$ such that $u(s'_k, -c_2(s'_k - t'_k) + \xi_{t'_k}) > 1 - \varepsilon_2$ for all $k \geq k_2$. One infers that $-c_2(s'_k - t'_k) + \xi_{t'_k} \leq \xi_{s'_k}$ for all $k \geq k_2$, which contradicts (2.18), since $t'_k < s'_k$. Hence, the proof of the lower bound (2.16) is complete.

Consequently, $\limsup_{|t-s|\to+\infty} |\xi_t - \xi_s|/|t - s| \leq C$, where the constant C defined in (2.9) depends only on the functions a and f. The proof of Theorem 1.4 is thereby complete. □

3 Uniqueness in the class of transition fronts

In this section, we prove Theorem 1.5 and Theorem 1.6 on the uniqueness of transition fronts for problem (1.1). Applying [10, Theorem 1.14 (1)] to the one-dimensional equation (1.1), one infers that a transition front u with positions $(\xi_t)_{t\in\mathbb{R}}$ is reduced to a pulsating front provided that there is a real number $c>0$ such that $\sup_{(t,s)\in\mathbb{R}^2}\big(|\xi_t-\xi_s|-c|t-s|\big)<+\infty$. Therefore, under the assumption of existence of a pulsating front with nonzero speed c_L, the conclusion of Theorem 1.5 follows easily by checking that for a given transition front u, the function $t\mapsto\xi_t-c_Lt$ is bounded. On the other hand, for the proof of Theorem 1.6, one cannot apply [10, Theorem 1.14 (1)] directly, since the existence of a pulsating front is not assumed a priori. Instead, we will use the sliding method with respect to the time variable to get the conclusion under the weaker assumption (1.13).

3.1 Proof of Theorem 1.5

The strategy for the proof of Theorem 1.5 is similar to that used in Proposition 2.2 of [20] for homogeneous equations. The main difference is that we will use the global stability and uniqueness of pulsating fronts for the spatially periodic equation (1.1), instead of these properties for standard traveling waves solving homogeneous equations. For the sake of completeness, we include all the details as follows.

Proof of Theorem 1.5. Let $u_L(t,x)=\phi_L(x-c_Lt,x/L)$ be a pulsating front of equation (1.1), in the sense of (1.5), with speed $c_L\neq 0$. Let $u(t,x)$ be a transition front connecting 0 and 1 for problem (1.1). That is, there is a family $(\xi_t)_{t\in\mathbb{R}}$ of real numbers such that u converges to the steady states 0 and 1 as $x\to\pm\infty$ in the sense of (1.4). As in the proof of Theorem 1.4, one can assume without loss of generality that $\xi_t\in L\mathbb{Z}$ for all $t\in\mathbb{R}$.

Now we prove that the function $t\mapsto\xi_t-c_Lt$ is bounded. Let $\overline{v}$ and $\underline{v}$ be the solutions of the Cauchy problem associated to (1.1), with initial conditions

$$\overline{v}(0,x)=\begin{cases}1 & \text{if } x\le 0,\\ \delta/2 & \text{if } x>0,\end{cases}\quad\text{and}\quad \underline{v}(0,x)=\begin{cases}1-\delta/2 & \text{if } x\le 0,\\ 0 & \text{if } x>0,\end{cases}$$

where $\delta\in(0,1/2)$ is the constant given in (1.2). Since the pulsating front ϕ_L is globally stable from Theorem 1.12 in [16], it follows that there exist two real numbers $\overline{\eta}$ and $\underline{\eta}$ such that

$$\sup_{y\in\mathbb{R}}\big|\overline{v}(s,y)-\phi_L(y-c_Ls+\overline{\eta},y)\big|+\sup_{y\in\mathbb{R}}\big|\underline{v}(s,y)-\phi_L(y-c_Ls+\underline{\eta},y)\big|\to 0\ \text{ as } s\to+\infty$$

(notice that only the assumption (1.2) is used in the proof of that stability result, while the assumption (1.11) on the profile of $f(x,\cdot)$ is not needed, see Proposition 4.4 in [16]). Therefore, since $\phi_L(-\infty,y)=1$ and $\phi_L(+\infty,y)=0$ uniformly in $y\in\mathbb{R}$, there are $T>0$ and $M>0$ such that, for all $s\ge T$,

$$\begin{cases}\underline{v}(s,y)>\delta/2 & \text{if } y\le c_Ls-M,\\ \overline{v}(s,y)<1-\delta/2 & \text{if } y\ge c_Ls+M.\end{cases}\tag{3.1}$$

On the other hand, from Definition 1.1, there is $B\in L\mathbb{N}$ such that

$$\forall\,(t,x)\in\mathbb{R}^2,\quad\begin{cases}x-\xi_t\ge B & \Longrightarrow\ 0<u(t,x)\le\delta/2,\\ x-\xi_t\le -B & \Longrightarrow\ 1-\delta/2\le u(t,x)<1,\end{cases}\tag{3.2}$$

whence $\underline{v}(0, x - \xi_{t_0} + B) \le u(t_0, x) \le \overline{v}(0, x - \xi_{t_0} - B)$ for all $x \in \mathbb{R}$ and $t_0 \in \mathbb{R}$. Then, since B and ξ_{t_0} belong to $L\mathbb{Z}$ for all $t_0 \in \mathbb{R}$ and since (1.1) is L-periodic in x, the maximum principle yields

$$\underline{v}(t - t_0, x - \xi_{t_0} + B) \le u(t, x) \le \overline{v}(t - t_0, x - \xi_{t_0} - B) \text{ for all } t \ge t_0 \text{ and } x \in \mathbb{R}.$$

This together with (3.1) implies that, for all $t_0 < t_0 + T \le t$,

$$\begin{cases} u(t, x) > \delta/2 & \text{if } x - \xi_{t_0} + B \le c_L(t - t_0) - M, \\ u(t, x) < 1 - \delta/2 & \text{if } x - \xi_{t_0} - B \ge c_L(t - t_0) + M. \end{cases}$$

Thus, from (3.2), one obtains that, for all $t_0 < t_0 + T \le t$,

$$\xi_{t_0} - B + c_L(t - t_0) - M < \xi_t + B \text{ and } \xi_{t_0} + B + c_L(t - t_0) + M > \xi_t - B. \tag{3.3}$$

By fixing $t = 0$, one gets $\limsup_{t_0 \to -\infty} |\xi_{t_0} - c_L t_0| \le |\xi_0| + 2B + M$. For any arbitrary $t \in \mathbb{R}$, letting $t_0 \to -\infty$ in (3.3) leads to $|\xi_t - c_L t| \le |\xi_0| + 4B + 2M$. Therefore, the function $t \mapsto \xi_t - c_L t$ is bounded.

Since the family of positions $(\xi_t)_{t \in \mathbb{R}}$ is defined up to an additive bounded function, without loss of generality, one can finally assume that $\xi_t = c_L t$ for all $t \in \mathbb{R}$. That is, ξ_t is monotone in t, and $|\xi_t - \xi_s| - |c_L||t - s| = 0$ for all $(t, s) \in \mathbb{R}^2$. It finally follows from [10, Theorem 1.14 (1)] that u is a pulsating front and that it is equal to u_L up to shift in time. The proof of Theorem 1.5 is thereby complete. □

3.2 Proof of Theorem 1.6

Let us now turn to the proof of Theorem 1.6. Due to the uniqueness result in Theorem 1.5, it is sufficient to prove the existence of a pulsating front with nonzero speed for equation (1.1). To do so, we first show that equation (1.1) admits a transition front v associated with a family $(\zeta_t)_{t \in \mathbb{R}}$ satisfying $\lim_{|t-s| \to +\infty} |\zeta_t - \zeta_s| = +\infty$. More precisely, we have the following result.

Lemma 3.1. *Under the assumptions of Theorem* 1.6, *equation* (1.1) *admits a transition front* $0 < v < 1$ *connecting* 0 *and* 1 *associated with a family* $(\zeta_t)_{t \in \mathbb{R}}$ *in the sense of* (1.4), *such that* $\lim_{|t-s| \to +\infty} |\zeta_t - \zeta_s| = +\infty$. *Furthermore, either there is a positive integer* K_1 *such that*

$$\zeta_{(k+1)K_1} > \zeta_{kK_1} + 1 \text{ for all } k \in \mathbb{Z}, \tag{3.4}$$

or there is a positive integer K_2 *such that*

$$\zeta_{(k+1)K_2} < \zeta_{kK_2} - 1 \text{ for all } k \in \mathbb{Z}. \tag{3.5}$$

Proof. Throughout the proof, we assume the existence of a transition front u connecting 0 and 1 for (1.1), associated with a family $(\xi_t)_{t \in \mathbb{R}}$ satisfying (1.4). We are also given a sequence $(t_n)_{n \in \mathbb{N}}$ satisfying (1.13). Firstly, as in the proof of Theorem 1.4, one can assume without loss of generality that $\xi_t \in L\mathbb{Z}$ for all $t \in \mathbb{R}$ and that ξ_t is constant on $[m, m+1)$ for all $m \in \mathbb{Z}$. Set

$$u_n(t, x) = u(t + t_n, x + \xi_{t_n}) \text{ for } (t, x) \in \mathbb{R}^2 \text{ and } n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, the function u_n satisfies (1.1). Furthermore, from Proposition 1.3, there holds

$$0 < \inf_{n \in \mathbb{N}} u_n(0, 0) \le \sup_{n \in \mathbb{N}} u_n(0, 0) < 1. \tag{3.6}$$

Up to extraction of some subsequence, one can assume, from standard parabolic estimates, that $u_n(t,x) \to v(t,x)$ as $n \to +\infty$ locally uniformly in $(t,x) \in \mathbb{R}^2$, where $0 \le v \le 1$ is a classical solution of (1.1). It follows from (3.6) that $0 < v(0,0) < 1$, whence $0 < v(t,x) < 1$ for all $(t,x) \in \mathbb{R}^2$ by the strong maximum principle.

On the other hand, for each $n \in \mathbb{N}$, the function $t \mapsto \xi_{t+t_n} - \xi_{t_n}$ ranges in $L\mathbb{Z}$ and is constant on $[m - t_n, m+1-t_n)$ for every $m \in \mathbb{Z}$. Furthermore, Lemma 2.1 implies that, for each $C \ge 0$, $\sup_{n\in\mathbb{N},\,|t|\le C} |\xi_{t+t_n} - \xi_{t_n}| < +\infty$. Therefore, up to extraction of a subsequence, there exists a function $t \mapsto \widetilde{\xi}_t$ such that $\widetilde{\xi}_t \in L\mathbb{Z}$ for all $t \in \mathbb{R}$ and $\xi_{t+t_n} - \xi_{t_n} \to \widetilde{\xi}_t$ as $n \to +\infty$ for every $t \in \mathbb{R}$.

Let us now show that v is a transition front connecting 0 and 1 for (1.1) with this family $(\widetilde{\xi}_t)_{t\in\mathbb{R}}$. To do so, let $\varepsilon > 0$ be any arbitrary positive real number. Let $M \ge 0$ be such that $u(t, x+\xi_t) \ge 1-\varepsilon$ for all $t \in \mathbb{R}$ and $x \le -M$, and $u(t,x+\xi_t) \le \varepsilon$ for all $t \in \mathbb{R}$ and $x \ge M$. For any $t \in \mathbb{R}$ and $x < \widetilde{\xi}_t - M$, one has

$$x + \xi_{t_n} - \xi_{t+t_n} \underset{n\to+\infty}{\longrightarrow} x - \widetilde{\xi}_t < -M,$$

whence $x + \xi_{t_n} < \xi_{t+t_n} - M$ for n large and $u_n(t,x) = u(t+t_n, x+\xi_{t_n}) \ge 1-\varepsilon$ for n large. As a consequence, $v(t,x) \ge 1-\varepsilon$ for all $t \in \mathbb{R}$ and $x < \widetilde{\xi}_t - M$. Similarly, one can show that $v(t,x) \le \varepsilon$ for all $t \in \mathbb{R}$ and $x > \widetilde{\xi}_t + M$. In other words, v is a transition front connecting 0 and 1, with the family $(\widetilde{\xi}_t)_{t\in\mathbb{R}}$ satisfying (1.4). Furthermore, by writing $\xi_{t+t_n} - \xi_{s+t_n} = (\xi_{t+t_n} - \xi_{t_n}) - (\xi_{s+t_n} - \xi_{t_n})$, it follows from assumption (1.13) that

$$\lim_{|t-s|\to+\infty} |\widetilde{\xi}_t - \widetilde{\xi}_s| = +\infty. \tag{3.7}$$

Let us set $\zeta_t = \widetilde{\xi}_k + (\widetilde{\xi}_{k+1} - \widetilde{\xi}_k)(t-k)$ for $t \in [k, k+1)$ and $k \in \mathbb{N}$. The family $(\zeta_t)_{t\in\mathbb{R}}$ is also associated to v in the sense of (1.4), since the function $t \mapsto \zeta_t - \widetilde{\xi}_t$ is bounded by Lemma 2.1 applied to v and $(\widetilde{\xi}_t)_{t\in\mathbb{R}}$. Moreover, by using again Lemma 2.1, the function $t \mapsto \zeta_t$ is actually Lipschitz continuous, and

$$\lim_{|t-s|\to+\infty} |\zeta_t - \zeta_s| = +\infty \tag{3.8}$$

from (3.7). In particular, there is a positive integer $K_0 > 0$ such that

$$|\zeta_{K_0 i} - \zeta_{K_0 j}| > 1 \ \text{ for all } i,\, j \in \mathbb{Z} \text{ with } i \neq j. \tag{3.9}$$

Next, we claim that either the set

$$E^+ := \big\{k \in \mathbb{N}\ ;\ \zeta_{K_0 k} > 0\big\} \cup \big\{k \in \mathbb{Z}\backslash\mathbb{N}\ ;\ \zeta_{K_0 k} < 0\big\}$$

or the set

$$E^- := \big\{k \in \mathbb{N}\ ;\ \zeta_{K_0 k} < 0\big\} \cup \big\{k \in \mathbb{Z}\backslash\mathbb{N}\ ;\ \zeta_{K_0 k} > 0\big\}$$

is finite. Suppose on the contrary that both E^+ and E^- are infinite. Since E^+ is infinite, let us assume here that the set $\big\{k \in \mathbb{N}\ ;\ \zeta_{K_0 k} > 0\big\}$ is infinite (the case where the set $\big\{k \in \mathbb{Z}\backslash\mathbb{N}\ ;\ \zeta_{K_0 k} < 0\big\}$ is infinite can be handled similarly). Since the set E^- is infinite, two cases may then occur.

Case 1: the set $\big\{k \in \mathbb{N}\ ;\ \zeta_{K_0 k} < 0\big\}$ *is infinite.* From (3.8), there exist two sequences of positive integers $(i_k)_{k\in\mathbb{N}}$ and $(j_k)_{k\in\mathbb{N}}$ such that

$$0 < i_0 < j_0 < i_1 < j_1 < \cdots < i_k < j_k < \cdots$$

and $\zeta_{K_0 j_{k+1}} < \zeta_{K_0 j_k} < 0 < \zeta_{K_0 i_k} < \zeta_{K_0 i_{k+1}}$ for all $k \in \mathbb{N}$. By continuity of the function $t \mapsto \zeta_t$, there exists a sequence of real numbers $(s_k)_{k\in\mathbb{N}}$ such that $K_0 i_k < s_k < K_0 j_k$ and $\zeta_{s_k} = 0$ for all $k \in \mathbb{N}$. The facts that $\lim_{k\to+\infty} s_k = +\infty$ and $\zeta_{s_k} = 0$ contradict (3.8). Therefore, case 1 is ruled out.

Case 2: the set $\{k \in \mathbb{Z}\backslash\mathbb{N} \ ; \ \zeta_{K_0 k} > 0\}$ *is infinite.* From (3.8), there exist then two sequences $(i'_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ and $(j'_k)_{k\in\mathbb{N}}$ in $\mathbb{Z}\backslash\mathbb{N}$ such that

$$\cdots < j'_{k+1} < j'_k < \cdots < j'_0 < 0 < i'_0 < \cdots < i'_k < i'_{k+1} < \cdots$$

and $\zeta_{K_0 i'_k} < \zeta_{K_0 j'_k} < \zeta_{K_0 i'_{k+1}} < \zeta_{K_0 j'_{k+1}}$ for all $k \in \mathbb{N}$. By continuity of $t \mapsto \zeta_t$, there exists a sequence of real numbers $(s'_k)_{k\in\mathbb{N}}$ such that $K_0 i'_k < s'_k < K_0 i'_{k+1}$ and $\zeta_{s'_k} = \zeta_{K_0 j'_k}$ for all $k \in \mathbb{N}$. But $s'_k \to +\infty$ while $K_0 j'_k \to -\infty$ as $k \to +\infty$, which contradicts (3.8). Thus, case 2 is ruled out too.

The proof of our claim is thereby complete, that is, either E^+ or E^- is a finite set. Finally, we prove that, if E^- is a finite set, then there is $K_1 \in K_0\mathbb{N}$ such that (3.4) holds, while there is $K_2 \in K_0\mathbb{N}$ such that (3.5) holds if E^+ is a finite set. We will just consider the case where the set E^- is finite, the proof for the case where the set E^+ is finite being similar. So, let us assume that the set E^- is finite. Even if it means increasing K_0, one can then assume without loss of generality that $\zeta_{K_0 k} \geq 0$ for all $k \in \mathbb{N}\backslash\{0\}$ and $\zeta_{K_0 k} \leq 0$ for all $k \in \mathbb{Z}\backslash\mathbb{N}$. As a consequence of (3.8), one infers that $\zeta_{K_0 k} \to \pm\infty$ as $k \to \pm\infty$ (and even $\zeta_t \to \pm\infty$ as $t \to \pm\infty$ by Lemma 2.1 applied to v and $(\zeta_t)_{t\in\mathbb{R}}$). Let us now assume by contradiction that (3.4) does not hold. In particular, for each $m \in \mathbb{N}\backslash\{0\}$, there is $k_m \in \mathbb{Z}$ such that $\zeta_{(k_m+1)K_0 m} \leq \zeta_{k_m K_0 m} + 1$, whence

$$\zeta_{(k_m+1)K_0 m} < \zeta_{k_m K_0 m} - 1$$

from (3.9). Since $t \mapsto \zeta_t$ is continuous and $\zeta_t \to +\infty$ as $t \to +\infty$, there are a sequence of real numbers $(\tau_m)_{m\in\mathbb{N}\backslash\{0\}}$ such that $\tau_m \geq (k_m+1)K_0 m$ and $\zeta_{\tau_m} = \zeta_{k_m K_0 m}$ for all $m \in \mathbb{N}\backslash\{0\}$. Since $\tau_m - k_m K_0 m \to +\infty$ as $m \to +\infty$, one gets a contradiction with (3.8).

Consequently, (3.4) holds if E^- is a finite set. Similarly, (3.5) holds if E^+ is a finite set. Since either E^- or E^+ is a finite set, it follows that either (3.4) or (3.5) holds and the proof of Lemma 3.1 is thereby complete. □

Let $0 < v < 1$ be the transition front given in Lemma 3.1 and associated with positions $(\zeta_t)_{t\in\mathbb{R}}$. By sliding v with respect to t, we will show that equation (1.1) admits a pulsating front with positive speed if $(\zeta_t)_{t\in\mathbb{R}}$ satisfies (3.4), while it admits a pulsating front with negative speed if $(\zeta_t)_{t\in\mathbb{R}}$ satisfies (3.5).

Lemma 3.2. *Assume that $(\zeta_t)_{t\in\mathbb{R}}$ satisfies* (3.4). *Then there is a real number $\tau_0 > 0$ such that $v(t+\tau, x+L) \geq v(t,x)$ for all $\tau \geq \tau_0$ and $(t,x) \in \mathbb{R}^2$.*

Proof. From Definition 1.1, there is $B > 0$ such that

$$\forall\,(t,x) \in \mathbb{R}^2, \quad \begin{cases} x - \zeta_t \geq B & \Longrightarrow \ 0 < v(t,x) \leq \delta, \\ x - \zeta_t \leq -B & \Longrightarrow \ 1 - \delta/2 \leq v(t,x) < 1, \end{cases} \tag{3.10}$$

where $\delta \in (0, 1/2)$ is the constant given in (1.2). As in the proof of Proposition 1.3, one can also assume that $f(x,\cdot)$ is (strictly) decreasing in $(-\infty,\delta]$ and $[1-\delta,+\infty)$ for every $x \in \mathbb{R}$. Because of (3.4) and Lemma 2.1 applied to v and $(\zeta_t)_{t\in\mathbb{R}}$, there is a real number $\tau_0 > 0$ such that $\zeta_{t+\tau} - \zeta_t \geq 2B + L$ for all $t \in \mathbb{R}$ and $\tau \geq \tau_0$. It then follows that, for any $\tau \geq \tau_0$,

$$v(t+\tau, x+L) \geq 1 - \delta \ \text{ for all } (t,x) \in \mathbb{R}^2 \text{ such that } x - \zeta_t \leq B. \tag{3.11}$$

Together with (3.10) and $1-\delta \geq \delta$, one gets that

$$v(t+\tau, x+L) \geq v(t,x) \text{ for all } (t,x) \in \mathbb{R}^2 \text{ with } x-\zeta_t = B. \tag{3.12}$$

Fix any $\tau \in [\tau_0, +\infty)$, define

$$\varepsilon^* := \inf\big\{\varepsilon > 0\ ;\ v(t+\tau, x+L)+\varepsilon \geq v(t,x) \text{ for all } (t,x) \in \mathbb{R}^2 \text{ with } x-\zeta_t \leq B\big\}$$

and let us show that $\varepsilon^* = 0$. Assume by contradiction that $\varepsilon^* > 0$. There are then a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive real numbers and a sequence $(t_n, x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^2$ such that $\varepsilon_n \to \varepsilon^*$ as $n \to +\infty$ and

$$v(t_n+\tau, x_n+L)+\varepsilon_n < v(t_n, x_n) \text{ and } x_n - \zeta_{t_n} \leq B \text{ for all } n \in \mathbb{N}. \tag{3.13}$$

By (3.12) and the uniform continuity of v in $\mathbb{R}^2$, one infers that $\sup_{n\in\mathbb{N}}(x_n - \zeta_{t_n}) < B$. Furthermore, since the map $t \mapsto \zeta_t$ is Lipschitz-continuous (that is true owing to the construction of ζ_t, but this property can always be assumed without loss of generality by Lemma 2.1), there is then $\rho > 0$ such that

$$\forall\, n \in \mathbb{N},\ \ \forall\,(t,x) \in \mathbb{R}^2,\ \ \big(|t-t_n|+|x-x_n| < \rho\big) \Longrightarrow \big(x-\zeta_t < B\big). \tag{3.14}$$

Write $x_n = x'_n + x''_n$ with $x'_n \in L\mathbb{Z}$, $x''_n \in [0, L)$ and assume that, up to extraction of a subsequence, $x''_n \to x_\infty \in [0, L]$ as $n \to +\infty$. Define $v_n(t,x) = v(t+t_n, x+x'_n)$ for $n \in \mathbb{N}$ and $(t,x) \in \mathbb{R}^2$. Up to extraction of a subsequence, the functions v_n converge locally uniformly in $\mathbb{R}^2$ to a solution $0 \leq v_\infty \leq 1$ of (1.1). Furthermore, for any $(t,x) \in \mathbb{R}^2$ with $|t|+|x| < \rho$, there holds $x + x_n - \zeta_{t+t_n} < B$ for all $n \in \mathbb{N}$ by (3.14), whence $v_n(t+\tau, x+x''_n+L) = v(t+t_n+\tau, x+x_n+L) \geq 1-\delta$ by (3.11) and

$$v_n(t+\tau, x+x''_n+L)+\varepsilon^* = v(t+t_n+\tau, x+x_n+L)+\varepsilon^* \geq v(t+t_n, x+x_n) = v_n(t, x+x''_n)$$

by definition of ε^*. Therefore, for any $(t,x) \in \mathbb{R}^2$ with $|t|+|x| < \rho$, one has

$$v_\infty(t+\tau, x+x_\infty+L) \geq 1-\delta \ \text{ and } \ v_\infty(t+\tau, x+x_\infty+L)+\varepsilon^* \geq v_\infty(t, x+x_\infty).$$

On the other hand, (3.13) implies that $v_n(\tau, x''_n+L)+\varepsilon_n < v_n(0, x''_n)$ for every $n \in \mathbb{N}$, whence $v_\infty(\tau, x_\infty+L)+\varepsilon^* \leq v_\infty(0, x_\infty)$ and finally $v_\infty(\tau, x_\infty+L)+\varepsilon^* = v_\infty(0, x_\infty)$. Finally, define $z(t,x) = v_\infty(t+\tau, x+L) - v_\infty(t,x)$ for $(t,x) \in \mathbb{R}^2$. There holds $z(t,x) \geq -\varepsilon^*$ for all $(t,x) \in \mathbb{R}^2$ such that $|t|+|x-x_\infty| < \rho$, while $z(0, x_\infty) = -\varepsilon^*$. Since z satisfies the equation $z_t - (a_L z_x)_x = f(x/L, v_\infty(t+\tau, x+L)) - f(x/L, v_\infty(t,x))$ in $\mathbb{R}^2$, one gets at $(0, x_\infty)$ that

$$f\Big(\frac{x_\infty}{L}, v_\infty(\tau, x_\infty+L)\Big) - f\Big(\frac{x_\infty}{L}, v_\infty(0, x_\infty)\Big) \leq 0. \tag{3.15}$$

However, $v_\infty(0, x_\infty) = v_\infty(\tau, x_\infty+L)+\varepsilon^* > v_\infty(\tau, x_\infty+L) \geq 1-\delta$ and $f(x_\infty/L, \cdot)$ is (strictly) decreasing in $[1-\delta, +\infty)$. Therefore, (3.15) is impossible, whence $\varepsilon^* = 0$ and $v(t+\tau, x+L) \geq v(t,x)$ for all $(t,x) \in \mathbb{R}^2$ with $x-\zeta_t \leq B$.

Similarly, by using the first property in (3.10) and (3.12), one gets that $v(t+\tau, x+L) \geq v(t,x)$ for all $\tau \geq \tau_0$ and for all $(t,x) \in \mathbb{R}^2$ with $x-\zeta_t \geq B$. The proof of Lemma 3.2 is thereby complete. □

Similarly, if $(\zeta_t)_{t\in\mathbb{R}}$ satisfies property (3.5), then there is a real number $\tau'_0 > 0$ such that $v(t+\tau, x) \leq v(t, x+L)$ for all $(t,x) \in \mathbb{R}^2$ and $\tau \geq \tau'_0$.

Based on the previous properties, we are now ready to prove Theorem 1.6.

Proof of Theorem 1.6. Firstly, assume that the family $(\zeta_t)_{t\in\mathbb{R}}$ satisfies (3.4). Then Lemma 3.2 yields the existence of $\tau_0 > 0$ such that $v(t+\tau, x+L) \geq v(t,x)$ for all $(t,x) \in \mathbb{R}^2$ and $\tau \geq \tau_0$. Define

$$\tau^* := \inf\big\{\tau > 0 \ ; \ v(t+\tau', x+L) \geq v(t,x) \text{ for all } (t,x) \in \mathbb{R}^2 \text{ and } \tau' \geq \tau\big\}.$$

One then has $0 \leq \tau^* \leq \tau_0$ and

$$v(t+\tau^*, x+L) \geq v(t,x) \ \text{ for all } (t,x) \in \mathbb{R}^2. \tag{3.16}$$

Now we claim that $\tau^* > 0$. Otherwise, $v(t, x+L) \geq v(t,x)$ for all $(t,x) \in \mathbb{R}^2$, whence $v(t, x+iL) \geq v(t,x)$ for all $(t,x) \in \mathbb{R}^2$ and $i \in \mathbb{N}$. Definition 1.1 applied to v implies in particular that $v(0,0) \leq \lim_{i\to+\infty} v(0, iL) = 0$, which contradicts the positivity of v. Therefore, $\tau^* > 0$.

Next, we prove that

$$\omega := \inf\big\{v(t+\tau^*, x+L) - v(t,x) \ ; \ |x-\zeta_t| \leq B\big\} = 0, \tag{3.17}$$

where $B > 0$ is as in (3.10). Assume, by contradiction, that $\omega > 0$. Notice that the derivative v_t is globally bounded from standard parabolic theory. Then there exists $\eta_0 \in (0, \tau^*)$ such that

$$v(t+\tau^*-\eta, x+L) \geq v(t,x) \ \text{ for all } \eta \in [0,\eta_0] \text{ and } |x-\zeta_t| \leq B. \tag{3.18}$$

For each $\eta \in [0,\eta_0]$, one then has $v(t+\tau^*-\eta, x+L) \geq v(t,x)$ for all $(t,x) \in \mathbb{R}^2$ with $x - \zeta_t = B$, while $0 < v(t,x) \leq \delta$ for all $(t,x) \in \mathbb{R}^2$ with $x - \zeta_t \geq B$ from (3.10). Therefore, with the same arguments as in the proof of Lemma 3.2, one infers that

$$v(t+\tau^*-\eta, x+L) \geq v(t,x) \ \text{ for all } \eta \in [0,\eta_0] \text{ and } x-\zeta_t \geq B. \tag{3.19}$$

On the other hand, since $v(t+\tau^*, x+L) \geq v(t,x) \geq 1-\delta/2$ for all $x - \zeta_t \leq -B$ by (3.10), one can assume without loss of generality that $\eta_0 > 0$ is small enough so that

$$v(t+\tau^*-\eta, x+L) \geq 1-\delta \ \text{ for all } \eta \in [0,\eta_0] \text{ and } x-\zeta_t \leq -B.$$ [1]

Since $v(t+\tau^*-\eta, x+L) \geq v(t,x)$ for all $(t,x) \in \mathbb{R}^2$ with $x - \zeta_t = -B$ from (3.18), one concludes as in Lemma 3.2 that

$$v(t+\tau^*-\eta, x+L) \geq v(t,x) \ \text{ for all } \eta \in [0,\eta_0] \text{ and } x-\zeta_t \leq -B.$$

This, together with (3.18)-(3.19), implies that $v(t+\tau^*-\eta, x+L) \geq v(t,x)$ for all $\eta \in [0,\eta_0]$ and $(t,x) \in \mathbb{R}^2$, contradicting the minimality of τ^*. Thus, the proof of (3.17) is finished.

We finally show that equation (1.1) admits a pulsating front with speed $c_L = L/\tau^* > 0$. From (3.17), there exists a sequence $(s_n, x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^2$ such that

$$v(s_n+\tau^*, x_n+L) - v(s_n, x_n) \to 0 \ \text{ as } \ n \to +\infty \ \text{ and } \ |x_n - \zeta_{s_n}| \leq B \text{ for all } n \in \mathbb{N}. \tag{3.20}$$

Let us now write $x_n = x_n' + x_n''$ with $x_n' \in L\mathbb{Z}$ and $x_n'' \in [0,L)$ and set

$$v_n(t,x) = v(t+s_n, x+x_n') \ \text{ for } (t,x) \in \mathbb{R}^2 \text{ and } n \in \mathbb{N}.$$

[1] Notice that this is the place where we use the choice of $\delta/2$ in the second property of (3.10).

For each $n \in \mathbb{N}$, the function v_n satisfies (1.1). Up to extraction of some subsequence, one can assume that $x''_n \to x_\infty \in [0, L]$ as $n \to +\infty$ and that, from standard parabolic estimates, $v_n(t, x) \to v_\infty(t, x)$ as $n \to +\infty$ locally uniformly in $\mathbb{R}^2$, where $0 \le v_\infty \le 1$ solves (1.1). Furthermore, $v_\infty(t+\tau^*, x+L) \ge v_\infty(t, x)$ for all $(t, x) \in \mathbb{R}^2$ from (3.16), and $v_\infty(\tau^*, x_\infty+L) = v_\infty(0, x_\infty)$ from (3.20). Since $v_\infty(t + \tau^*, x + L)$ is also a solution to equation (1.1), it follows from the strong parabolic maximum principle that

$$v_\infty(t + \tau^*, x + L) = v_\infty(t, x) \text{ for all } (t, x) \in \mathbb{R}^2.$$

On the other hand, from Definition 1.1, one has

$$v_n(t, x) = v(t + s_n, x + x'_n) \to 0 \ \text{ as } x + x'_n - \zeta_{s_n+t} \to +\infty$$

uniformly with respect to $n \in \mathbb{N}$ and $(t, x) \in \mathbb{R}^2$. Write

$$x + x'_n - \zeta_{s_n+t} = (x - \zeta_t) + (x_n - \zeta_{s_n}) - x''_n + (\zeta_t + \zeta_{s_n} - \zeta_{t+s_n})$$

and remember that the sequences $(x''_n)_{n\in\mathbb{N}}$ and $(x_n - \zeta_{s_n})_{n\in\mathbb{N}}$ are bounded. Notice that, thanks to Lemma 2.1, the quantities ζ_t and $\zeta_t + \zeta_{s_n} - \zeta_{t+s_n}$ are bounded locally in $t \in \mathbb{R}$ and independently of $n \in \mathbb{N}$. It then follows that

$$v_\infty(t, x) \to 0 \ \text{ as } x \to +\infty \text{ locally in } t \in \mathbb{R}.$$

Similar arguments imply that $v_\infty(t, x) \to 1$ as $x \to -\infty$ locally in $t \in \mathbb{R}$. Therefore, v_∞ is a pulsating front with positive speed $c_L = L/\tau^*$. Thus, from Theorem 1.5, any transition front connecting 0 and 1 for equation (1.1) is equal to v_∞ up to shift in time.

Similarly, if $(\zeta_t)_{t\in\mathbb{R}}$ satisfies property (3.5), then one can conclude that equation (1.1) admits a pulsating front with speed $\widetilde{c}_L = -L/\tau_* < 0$, where τ_* is the positive number defined by

$$\tau_* = \inf\big\{\tau > 0\ ;\ v(t + \tau', x) \le v(t, x + L) \text{ for all } (t, x) \in \mathbb{R}^2 \text{ and } \tau' \ge \tau\big\}.$$

Furthermore, any transition front connecting 0 and 1 is equal to this pulsating front up to shift in time. The proof of Theorem 1.6 is thereby complete. $\square$

4 Transition fronts which are not pulsating fronts

This section is devoted to the proof of Theorem 1.7 on the existence of a new type of transition fronts, which are not pulsating fronts. Let us explain the general strategy before going into the details. The first step consists in constructing super- and subsolutions for equation (1.1) when the period L is large. The construction will use the properties of the standard traveling wave for the homogeneous equation

$$(u^y)_t(t, x) = a\,(u^y)_{xx}(t, x) + f^y(u^y(t, x))$$

with $0 < u^y(t, x) < 1$ for all $(t, x) \in \mathbb{R}^2$, where y is a fixed real number and $f^y = f(y, \cdot)$ in $[0, 1]$ (remember that a is a constant in Theorem 1.7). As a matter of fact, for large L, by choosing some fixed suitable real numbers $\overline{y} > \underline{y}$, we will get a supersolution $\overline{u}_L$ and a subsolution $\underline{u}_L$ of (1.1) which are located around the positions $L\overline{y}$ and $L\underline{y}$ respectively, which are strictly ordered and which move towards each other as time runs. As a consequence, there exist stationary fronts for equation (1.1) at large L, which emerge from these super- and subsolutions. By shifting suitably in space the subsolution (or the supersolution) of the first step, we can then construct a pair of strictly ordered stationary fronts such that the lower one is stable from above, and the upper one is stable from below. Once this is done, the Dancer-Hess connecting orbit lemma will lead to the existence of time-monotone transition fronts between these two stationary fronts. Actually, we will see that there exist at least two transition fronts: the first one is decreasing in time and the second one is increasing. Finally, we will prove that all transition fronts have zero global mean speed. In Section 4.1, we provide a series of auxiliary lemmas for the proof of Theorem 1.7, which is carried out in Section 4.2.

4.1 Some auxiliary lemmas

We recall that a, $\partial_u f(\cdot,0)$ and $\partial_u f(\cdot,1)$ are constants. Denote $\mu_+ = \partial_u f(\cdot,0)$ and $\mu_- = \partial_u f(\cdot,1)$, and remember that $\max(\mu_+,\mu_-) \le -\gamma < 0$ where $\gamma > 0$ is given in (1.2). Since the function $f(x,u)$ is continuous in $\mathbb{R}\times[0,1]$ and 1-periodic in x, one can choose $\overline{x} \in [0,1]$ such that

$$\int_0^1 f(\overline{x},u)du = \min_{x\in\mathbb{R}} \int_0^1 f(x,u)du.$$

Due to the assumptions (1.2) and (1.11), the profile $f(\overline{x},\cdot) : [0,1] \mapsto \mathbb{R}$ is bistable in the sense of (1.8). It then follows from [4, 19] that the homogeneous equation

$$u_t(t,x) = a\,u_{xx}(t,x) + f(\overline{x},u(t,x)) \tag{4.1}$$

admits a standard traveling front $\overline{\phi}(x-\overline{c}t)$ such that $0<\overline{\phi}<1$ in $\mathbb{R}$, $\overline{\phi}(-\infty)=1$ and $\overline{\phi}(+\infty)=0$. The front $\overline{\phi}(x)$ is decreasing in x and is unique up to shift in x (one can normalize $\overline{\phi}$ in such a way that $\overline{\phi}(0)=1/2$). Furthermore, the speed $\overline{c}$ is unique and it has the sign of the integral $\int_0^1 f(\overline{x},u)du$, that is, $\overline{c} < 0$ thanks to the assumption (1.15). Moreover, $\overline{\phi}(x)$ approaches the limits 1 and 0 as $x\to\pm\infty$ exponentially, namely there are two positive constants $A_\pm$ such that

$$\overline{\phi}(x) \sim A_+ \mathrm{e}^{-\overline{\lambda}_+ x} \text{ as } x\to+\infty,\ \ 1-\overline{\phi}(x) \sim A_- \mathrm{e}^{\overline{\lambda}_- x} \text{ as } x\to-\infty,\ \ \overline{\lambda}_\pm = \frac{\pm\overline{c}+\sqrt{\overline{c}^2-4a\mu_\pm}}{2a} > 0. \tag{4.2}$$

Lastly, $\overline{\phi}'(x)/\overline{\phi}(x) \to -\overline{\lambda}_+$ as $x\to+\infty$ and $\overline{\phi}'(x)/(1-\overline{\phi}(x)) \to -\overline{\lambda}_-$ as $x\to-\infty$.

Lemma 4.1. *There is $L_1>0$ large enough such that for every $L>L_1$, the function*

$$\overline{u}_L(x) = \overline{\phi}(x-L\overline{x})$$

is a strict supersolution of (1.1).

Proof. Fix any real number ε such that $0<\varepsilon<\min(-\overline{c}\,\overline{\lambda}_+/3, -\overline{c}\,\overline{\lambda}_-/3)$ (remember that $\overline{c}<0$). Since $f(\cdot,0) = f(\cdot,1) = 0$ and since $\partial_u f$ is continuous in $\mathbb{R}\times[0,1]$ and 1-periodic in x with $\mu_+ = \partial_u f(\cdot,0)$ and $\mu_- = \partial_u f(\cdot,1)$ being constant, it follows that there exists a number $C>0$ (independent of $L>0$) such that

$$(\mu_+-\varepsilon)\overline{\phi}(x) \le f(y,\overline{\phi}(x)) \le (\mu_++\varepsilon)\overline{\phi}(x) \ \text{ for all } x\ge C \text{ and } y\in\mathbb{R}, \tag{4.3}$$

$$-(\mu_-+\varepsilon)(1-\overline{\phi}(x)) \le f(y,\overline{\phi}(x)) \le -(\mu_--\varepsilon)(1-\overline{\phi}(x)) \ \text{ for all } x\le -C \text{ and } y\in\mathbb{R}, \tag{4.4}$$

and

$$\frac{\overline{c}\,\overline{\phi}'(x)}{\overline{\phi}(x)}+\overline{c}\,\overline{\lambda}_+\ge-\varepsilon\ \text{ for all } x\ge C, \tag{4.5}$$

$$\frac{\overline{c}\,\overline{\phi}'(x)}{1-\overline{\phi}(x)}+\overline{c}\,\overline{\lambda}_-\ge-\varepsilon\ \text{ for all } x\le -C. \tag{4.6}$$

Let us now check that $N_L(x):=-a\,\overline{u}_L''(x)-f_L(x,\overline{u}_L(x))>0$ in $\mathbb{R}$ for all $L>0$ large enough. Since $\overline{\phi}(x-\overline{c}t)$ is a solution to (4.1), one infers that, for all $x\in\mathbb{R}$,

$$N_L(x)=\overline{c}\,\overline{\phi}'(x-L\overline{x})+f\big(\overline{x},\overline{\phi}(x-L\overline{x})\big)-f_L\big(x,\overline{\phi}(x-L\overline{x})\big). \tag{4.7}$$

If $x\ge C+L\overline{x}$, then from (4.3), (4.5) and the choice of ε, one has

$$\begin{aligned}N_L(x)&\ge\overline{c}\,\overline{\phi}'(x-L\overline{x})+(\mu_+-\varepsilon)\overline{\phi}(x-L\overline{x})-(\mu_++\varepsilon)\overline{\phi}(x-L\overline{x})\\&=\overline{\phi}(x-L\overline{x})\Big(\frac{\overline{c}\,\overline{\phi}'(x-L\overline{x})}{\overline{\phi}(x-L\overline{x})}-2\varepsilon\Big)\ge\overline{\phi}(x-L\overline{x})(-\overline{c}\,\overline{\lambda}_+-3\varepsilon)\ >\ 0.\end{aligned}$$

On the other hand, if $x\le -C+L\overline{x}$, then, from (4.4), (4.6) and the choice of ε, there holds

$$\begin{aligned}N_L(x)&\ge\overline{c}\,\overline{\phi}'(x-L\overline{x})-(\mu_-+\varepsilon)\big(1-\overline{\phi}(x-L\overline{x})\big)+(\mu_--\varepsilon)\big(1-\overline{\phi}(x-L\overline{x})\big)\\&=\big(1-\overline{\phi}(x-L\overline{x})\big)\Big(\frac{\overline{c}\,\overline{\phi}'(x-L\overline{x})}{1-\overline{\phi}(x-L\overline{x})}-2\varepsilon\Big)\ge\big(1-\overline{\phi}(x-L\overline{x})\big)(-\overline{c}\,\overline{\lambda}_--3\varepsilon)\ >\ 0.\end{aligned}$$

Finally, for $-C+L\overline{x}\le x\le C+L\overline{x}$, due to the continuity and negativity of $\overline{\phi}'$, one gets that $\overline{\phi}'(x-L\overline{x})\le-\beta<0$ for some positive real number $\beta:=-\max_{[-C,C]}\phi'>0$ which is independent of $L>0$. Since the function $f(x,u)$ is of class $C^{0,\alpha}$ in x uniformly in $u\in[0,1]$, one concludes that there is a positive real number κ independent of $L>0$ such that, for all $-C+L\overline{x}\le x\le C+L\overline{x}$,

$$\big|f\big(\overline{x},\overline{\phi}(x-L\overline{x})\big)-f_L\big(x,\overline{\phi}(x-L\overline{x})\big)\big|\le\kappa\Big|\overline{x}-\frac{x}{L}\Big|^{\alpha}\le\frac{\kappa\,C^{\alpha}}{L^{\alpha}},$$

whence $N_L(x)\ge-\overline{c}\beta-\kappa C^{\alpha}/L^{\alpha}$ from (4.7) and $\overline{c}<0$. Therefore, if $L>L_1:=(\kappa C^{\alpha}/(|\overline{c}|\beta))^{1/\alpha}>0$, then $N_L(x)>0$ for all $-C+L\overline{x}\le x\le C+L\overline{x}$.

Thus, for every $L>L_1$, one has $N_L(x)>0$ for all $x\in\mathbb{R}$. Namely, $\overline{u}_L$ is a strict supersolution of equation (1.1). The proof of Lemma 4.1 is thereby compete. □

A subsolution for equation (1.1) at large L can be constructed in a similar way. More precisely, choose $\underline{x}\in[-1,0]$ such that

$$\int_0^1 f(\underline{x},u)du=\max_{x\in\mathbb{R}}\int_0^1 f(x,u)du.$$

It follows from (1.11) and (1.15) that, for the homogeneous equation $v_t=a\,v_{xx}+f(\underline{x},v)$, there are a unique $\underline{c}>0$ and a standard traveling front $v(t,x)=\underline{\phi}(x-\underline{c}t)$ such that $0<\underline{\phi}<1$ in $\mathbb{R}$, $\underline{\phi}(-\infty)=1$ and $\underline{\phi}(+\infty)=0$. The front $\underline{\phi}$ is decreasing in $\mathbb{R}$ and is unique up to shifts. Furthermore, there are two positive constants $B_\pm$ such that

$$\underline{\phi}(x)\sim B_+\mathrm{e}^{-\underline{\lambda}_+x}\text{ as } x\to+\infty,\ \ 1-\underline{\phi}(x)\sim B_-\mathrm{e}^{\underline{\lambda}_-x}\text{ as } x\to-\infty,\ \ \underline{\lambda}_\pm=\frac{\pm\underline{c}+\sqrt{\underline{c}^2-4a\mu_\pm}}{2a}>0. \tag{4.8}$$

Lemma 4.2. *There is $L_2 > 0$ large enough such that for every $L > L_2$, the function*

$$\underline{u}_L(x) = \underline{\phi}(x - L\underline{x})$$

is a strict subsolution of equation (1.1).

Proof. The proof is similar to that of Lemma 4.1, and we omit the details. □

Furthermore, when L is large enough, the following comparison holds.

Lemma 4.3. *There is $L^* > 0$ large enough such that for every $L > L^*$, $0 < \underline{u}_L < \overline{u}_L < 1$ in $\mathbb{R}$.*

Proof. From (4.2), (4.8) and $\overline{c} < 0 < \underline{c}$, one infers that $0 < \overline{\lambda}_+ < \underline{\lambda}_+$ and $\overline{\lambda}_- > \underline{\lambda}_- > 0$, whence $\overline{\phi}(x)/\underline{\phi}(x) \to +\infty$ as $x \to +\infty$ and $(1 - \overline{\phi}(x))/(1 - \underline{\phi}(x)) \to 0$ as $x \to -\infty$. It then follows that there is $C' > 0$ such that $\overline{\phi}(x) > \underline{\phi}(x)$ for all $x \in \mathbb{R}\backslash[-C', C']$. Remember that $\overline{\phi}$ and $\underline{\phi}$ are decreasing in $\mathbb{R}$ and that $-1 \le \underline{x} \le 0 \le \overline{x} \le 1$ with $\underline{x} \ne \overline{x}$, whence $\underline{x} < \overline{x}$. Consequently, there is $L^* > 0$ large enough such that, for every $L > L^*$ and $x \in \mathbb{R}$, $1 > \overline{\phi}(x - L\overline{x}) > \underline{\phi}(x - L\underline{x}) > 0$. The proof of Lemma 4.3 is thereby complete. □

Without loss of generality, one can assume that $L^* > \max(L_1, L_2)$, where L_1 and L_2 are given in Lemmas 4.1 and 4.2. Thus, for any given $L > L^*$, $0 < \underline{u}_L < \underline{u}_L < 1$ are strictly ordered sub- and supersolutions for equation (1.1), which both converge to 0 and 1 as $x \to \pm\infty$. As a consequence of the maximum principle, as will be explained in the proof of Theorem 1.7 below, stationary fronts for (1.1) will then exist, that is, stationary solutions $0 < p(x) < 1$ such that $p(-\infty) = 1$ and $p(+\infty) = 0$.

Before going further on, we need to prove a useful property, which actually holds for any $L > 0$, on the asymptotic behavior at $\pm\infty$ of any stationary front of (1.1).

Lemma 4.4. *Let $L > 0$ be arbitrary and let $0 < p(x) < 1$ be a stationary solution of equation* (1.1) *such that $p(-\infty) = 1$ and $p(+\infty) = 0$. Then there are two positive constants $M_\pm$ such that*

$$p(x) \sim M_+ \mathrm{e}^{-\lambda_+ x} \text{ as } x \to +\infty, \quad 1 - p(x) \sim M_- \mathrm{e}^{\lambda_- x} \text{ as } x \to -\infty, \quad \lambda_\pm = \sqrt{\frac{-\mu_\pm}{a}} > 0. \tag{4.9}$$

Proof. We will just show the first property of (4.9), that is, $p(x)$ decays exponentially to 0 as $x \to +\infty$, the proof of the behavior as $x \to -\infty$ being similar. First of all, because of (1.2), there is $\widetilde{x} \in \mathbb{R}$ such that $f_L(x, p(x)) < 0$ for all $x \ge \widetilde{x}$, whence $p''(x) > 0$ for all $x \ge \widetilde{x}$. Thus, p' is increasing in $[\widetilde{x}, +\infty)$ and since $p(+\infty) = 0$, one infers that $p'(+\infty) = 0$ and $p' < 0$ in $[\widetilde{x}, +\infty)$. On the other hand, from standard elliptic interior estimates and from Harnack inequality, the function p'/p is bounded in $\mathbb{R}$. Hence $\lambda := \limsup_{x\to+\infty} p'(x)/p(x) \in (-\infty, 0]$.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ such that $x_n \to +\infty$ and $p'(x_n)/p(x_n) \to \lambda$ as $n \to +\infty$. Write $x_n = x'_n + x''_n$ with $x'_n \in L\mathbb{Z}$ and $x''_n \in [0, L)$. Without loss of generality, one can assume that $x'_n \ge \widetilde{x}$ for all $n \in \mathbb{N}$ and that $x''_n \to y \in [0, L]$ as $n \to +\infty$. From the boundedness of p'/p, the functions

$$x \mapsto p_n(x) := \frac{p(x + x'_n)}{p(x'_n)} > 0$$

are locally bounded (that is, $\sup_{n\in\mathbb{N}} \|p_n\|_{L^\infty(K)} < +\infty$ for any compact subset $K \subset \mathbb{R}$). Furthermore, since equation (1.1) is L-periodic in x, the functions p_n obey $a\, p''_n + f_L(x, p(x'_n)p_n)/p(x'_n) = 0$ in $\mathbb{R}$.

Notice that $p(x'_n) \to 0$ as $n \to +\infty$ since $x'_n \to +\infty$, and remember that $\partial_u f$ is uniformly continuous in $\mathbb{R} \times [0,1]$ and that $\partial_u f(\cdot,0) = \mu_+ < 0$ is constant. From standard elliptic estimates, there is then a nonnegative function p_∞ such that, up to extraction of a subsequence, $p_n \to p_\infty$ in $C^1_{loc}(\mathbb{R})$ and p_∞ is a classical solution of $a\,p''_\infty + \mu_+ p_\infty = 0$ in $\mathbb{R}$. Furthermore, $p_\infty(0) = 1$, whence $p_\infty > 0$ in $\mathbb{R}$ from the strong maximum principle. On the other hand, since $x'_n \geq \widetilde{x}$ for all $n \in \mathbb{N}$ and $p' < 0$ on $[\widetilde{x}, +\infty)$, there holds $p'_n < 0$ on $[0, +\infty)$, whence $p'_\infty \leq 0$ in $[0,+\infty)$ (actually, $p'_\infty \leq 0$ in $\mathbb{R}$ since $x'_n \to +\infty$ as $n \to +\infty$). Therefore, $p_\infty(x) = e^{-\lambda_+ x}$ for all $x \in \mathbb{R}$, with $\lambda_+ = \sqrt{-\mu_+/a} > 0$. Finally, since $p'_n(x''_n)/p_n(x''_n) = p'(x_n)/p(x_n) \to \lambda$ as $n \to +\infty$, one infers that $p'_\infty(y)/p_\infty(y) = \lambda$, whence $\lambda = -\lambda_+$.

One has then proved that $\limsup_{x\to+\infty} p'(x)/p(x) = -\lambda_+$. With similar arguments, one can get that $\liminf_{x\to+\infty} p'(x)/p(x) = -\lambda_+$, and finally $p'(x)/p(x) \to -\lambda_+$ as $x \to +\infty$. Then the asymptotic stability theory for ordinal differential equations (see, e.g. [15, Chapter 13, Theorem 4.5]) implies that $p(x) \sim M_+ \mathrm{e}^{-\lambda_+ x}$ as $x \to +\infty$ for some constant $M_+ > 0$.

Similarly, applying the above analysis to the function $1-p$, one concludes that $1-p(x) \sim M_- \mathrm{e}^{\lambda_- x}$ as $x \to -\infty$ for some constant $M_- > 0$. The proof of Lemma 4.4 is thereby complete. □

In the following lemma, we show that when $L \geq L^*$, any transition front connecting 0 and 1 for problem (1.1) can be bounded from above by a translate of $\overline{u}_L$ and from below by a translate of $\underline{u}_L$ at large time, up to some exponentially small terms. As a matter of fact, this property will be used in Theorem 1.7 to prove that all transition fronts have zero global mean speed.

Lemma 4.5. *Even if it means increasing $L^* > 0$, the following property holds: for every $L > L^*$ and every transition front $0 < u < 1$ connecting 0 and 1 for problem (1.1), there are some real numbers x_*, x^*, $q_0 > 0$ and $\gamma_0 > 0$ such that*

$$\max\big(\underline{u}_L(x - x_*) - q_0 \mathrm{e}^{-\gamma_0 t}, 0\big) \leq u(s+t, x+\xi_s) \leq \min\big(\overline{u}_L(x - x^*) + q_0 \mathrm{e}^{-\gamma_0 t}, 1\big) \tag{4.10}$$

for all $x \in \mathbb{R}$, $s \in \mathbb{R}$ and $t \geq 0$. Furthermore, q_0 and γ_0 can be chosen independently of L and u.

Proof. We will only prove the upper inequality of (4.10), since the proof of the lower one is similar. We are given a transition front u connecting 0 and 1 for (1.1), for some $L > 0$ (that will be large at the end of the proof), associated with a family $(\xi_t)_{t\in\mathbb{R}}$ satisfying (1.4). First of all, as already emphasized, one can assume without loss of generality that $\xi_s \in L\mathbb{Z}$ for all $s \in \mathbb{R}$.

We first claim that, for any real number $q_0 \in (0,1)$, there exists a $\eta_0 \in L(\mathbb{N}\backslash\{0\})$ such that

$$u(s, x+\xi_s) \leq \overline{u}_L(x - \eta_0) + q_0 \ \text{ for all } (s,x) \in \mathbb{R}^2. \tag{4.11}$$

Indeed, from Definition 1.1, for any $q_0 \in (0,1)$, there is $M > 0$ such that $u(s, x+\xi_s) \leq q_0$ for all $x \geq M$ and $s \in \mathbb{R}$. Since $\lim_{x\to-\infty} \overline{u}_L(x) = 1$ and $0 < u < 1$ in $\mathbb{R}^2$, there exists $\eta_0 \in L(\mathbb{N}\backslash\{0\})$ large enough such that $u(s+\xi_s, x) \leq \overline{u}_L(x-\eta_0) + q_0$ for all $x \leq M$ and $s \in \mathbb{R}$. Therefore, the inequality (4.11) is proved.

Next, we set

$$v_L(t,x) := \min\big(\overline{u}_L(x - \eta(t)) + q(t), 1\big) > 0 \ \text{ for } t > 0 \text{ and } x \in \mathbb{R},$$

where η and q are $C^1([0,+\infty))$ functions such that $\eta(0) = \eta_0$, $\eta'(t) > 0$ for all $t \geq 0$, $q(0) = q_0$ and $0 < q(t) \leq q_0$ for all $t \geq 0$. By choosing later some appropriate functions $\eta(t)$ and $q(t)$, we will show that $v_L(t,x)$ is a supersolution of equation (1.1). To do so, for all $(t,x)\in(0,+\infty)\times\mathbb{R}$ such that $v_L(t,x)<1$, define

$$N(t,x):=\partial_t v_L(t,x)-a\partial_{xx}v_L(t,x)-f_L\big(x,v_L(t,x)\big).$$

A straightforward calculation gives, for all $(t,x)\in(0,+\infty)\times\mathbb{R}$ such that $v_L(t,x)<1$,

$$N(t,x)=q'(t)-\eta'(t)\overline{u}'_L(x-\eta(t))-a\overline{u}''_L(x-\eta(t))-f_L\big(x,\overline{u}_L(x-\eta(t))+q(t)\big).$$

Since $\overline{u}_L(x)=\overline{\phi}(x-L\overline{x})$ and $\overline{\phi}(x-\overline{c}t)$ is a traveling front of equation (4.1), it follows that, for all $(t,x)\in(0,+\infty)\times\mathbb{R}$ such that $v_L(t,x)<1$,

$$N(t,x)=\big(\overline{c}-\eta'(t)\big)\overline{u}'_L(x-\eta(t))+q'(t)+f\big(\overline{x},\overline{u}_L(x-\eta(t))\big)-f_L\big(x,\overline{u}_L(x-\eta(t))+q(t)\big).$$

For all such (t,x), define

$$\begin{cases} N_1(t,x):=-\eta'(t)\overline{u}'_L(x-\eta(t))+q'(t)+f_L\big(x,\overline{u}_L(x-\eta(t))\big)-f_L\big(x,\overline{u}_L(x-\eta(t))+q(t)\big),\\ N_2(t,x):=\overline{c}\,\overline{u}'_L(x-\eta(t))+f\big(\overline{x},\overline{u}_L(x-\eta(t))\big)-f_L\big(x,\overline{u}_L(x-\eta(t))\big),\end{cases}$$

so that $N(t,x)=N_1(t,x)+N_2(t,x)$.

In this paragraph, we will choose suitable functions $q(t)$ and $\eta(t)$ such that $N_1(t,x)\geq 0$ for all $(t,x)\in(0,+\infty)\times\mathbb{R}$ such that $v_L(t,x)<1$. Since f satisfies (1.2) and $\partial_u f(x,u)$ is continuous in $\mathbb{R}\times[0,1]$ and 1-periodic in x, there exists a real number $\delta_0\in(0,1)$ (which depends only on f) such that, if $(q_0,t,x)\in(0,\delta_0]\times(0,+\infty)\times\mathbb{R}$ with $\overline{u}_L(x-\eta(t))\in[0,\delta_0]\cup[1-\delta_0,1]$ and $v_L(t,x)<1$, then $f_L\big(x,\overline{u}_L(x-\eta(t))\big)-f_L\big(x,\overline{u}_L(x-\eta(t))+q(t)\big)\geq(\gamma/2)q(t)$ and

$$N_1(t,x)\geq-\eta'(t)\overline{u}'_L(x-\eta(t))+q'(t)+\frac{\gamma q(t)}{2}. \tag{4.12}$$

On the other hand, due to the monotonicity of $\overline{\phi}$, there is a constant $\rho=-\max_{\delta_0\leq\overline{\phi}(y)\leq 1-\delta_0}\overline{\phi}'(y)>0$ (which actually depends only on $\overline{\phi}$ and δ_0, that is, only on f) such that if $(t,x)\in(0,+\infty)\times\mathbb{R}$ with $\overline{u}_L(x-\eta(t))\in[\delta_0,1-\delta_0]$ and $v_L(t,x)<1$, then $\overline{u}'_L(x-\eta(t))=\overline{\phi}'(x-\eta(t)-L\overline{x})\leq-\rho$, whence

$$N_1(t,x)\geq\rho\,\eta'(t)+q'(t)-Kq(t), \tag{4.13}$$

where $K=\max_{(x,u)\in\mathbb{R}\times[0,1]}|\partial_u f(x,u)|$. Now, let us choose $q_0=\delta_0$ (depending only on f). There exists $\eta_0\in L(\mathbb{N}\backslash\{0\})$ (depending on L and u) such that (4.11) holds. Let us then choose $q(t)$ and $\eta(t)$ such that

$$q(0)=q_0,\ q'(t)=-\frac{\gamma q(t)}{2}\text{ for all }t\geq 0,\ \eta(0)=\eta_0\text{ and }\eta'(t)=\frac{2K+\gamma}{2\rho}q(t)\text{ for all }t\geq 0. \tag{4.14}$$

Namely, $q(t)=q_0\mathrm{e}^{-\gamma t/2}$ and $\eta(t)=\eta_0+q_0(2K+\gamma)(1-\mathrm{e}^{-\gamma t/2})/(\gamma\rho)$ for all $t\geq 0$. It is easy to check from (4.12)-(4.14) and the negativity of $\overline{u}'_L$ that $q(t)$ and $\eta(t)$ are such that $N_1(t,x)\geq 0$ for all $(t,x)\in(0,+\infty)\times\mathbb{R}$ such that $v_L(t,x)<1$. Notice also that the functions q and $\eta-\eta_0$ depend only on f and are thus independent of L and u (but η_0 and $\eta(t)$ depend on L and u)

Now, for those chosen functions $q(t)$ and $\eta(t)$, we show that $N_2(t,x)\geq 0$ for all $(t,x)\in(0,+\infty)\times\mathbb{R}$ such that $v_L(t,x)<1$, as soon as L is large enough. The arguments are similar to those used in the proof of Lemma 4.1. More precisely, since $\overline{u}_L(x-\eta(t))=\overline{\phi}(x-\eta(t)-L\overline{x})$, it follows that, for all $(t,x)\in(0,+\infty)\times\mathbb{R}$ such that $v_L(t,x)<1$,

$$N_2(t,x)=\overline{c}\,\overline{\phi}'(x-\eta(t)-L\overline{x})+f\big(\overline{x},\overline{\phi}(x-\eta(t)-L\overline{x})\big)-f_L\big(x,\overline{\phi}(x-\eta(t)-L\overline{x})\big),$$

that is, $N_2(t,x)$ is nothing but the expression (4.7) with $x-\eta(t)$ instead of x. As a consequence, as in the proof of Lemma 4.1, there exists a positive number C (independent of L and u) such that $N_2(t,x)>0$ for all $(t,x)\in(0,+\infty)\times\mathbb{R}$ with $|x-\eta(t)-L\overline{x}|\geq C$ and $v_L(t,x)<1$. On the other hand, there is a constant $\beta>0$ (independent of L and u) such that $\overline{\phi}'(x-\eta(t)-L\overline{x})\leq-\beta$ for all $(t,x)\in(0,+\infty)\times\mathbb{R}$ with $|x-\eta(t)-L\overline{x}|\leq C$ and $v_L(t,x)<1$. Moreover, in this case, since

$$x-\eta_0-L\overline{x}=(x-\eta(t)-L\overline{x})+(\eta(t)-\eta_0)$$

and since the function $t\mapsto\eta(t)-\eta_0$ is bounded and independent of L and u, there exists a positive real number C_1 (independent of L, u and (t,x)) such that $|x-\eta_0-L\overline{x}|\leq C_1$ if $|x-\eta(t)-L\overline{x}|\leq C$. Remember that $f(x,u)$ is 1-periodic in x, that $f(x,u)$ is of class $C^{0,\alpha}$ in x uniformly in $u\in[0,1]$, that $\eta_0\in L\mathbb{Z}$ and that $\overline{c}<0$. One then concludes that, for all $(t,x)\in(0,+\infty)\times\mathbb{R}$ such that $|x-\eta(t)-L\overline{x}|\leq C$ and $v_L(t,x)<1$, there holds

$$N_2(t,x)=\overline{c}\,\overline{\phi}'(x-\eta(t)-\overline{x}L)+f\big(\overline{x},\overline{\phi}(x-\eta(t)-\overline{x}L)\big)-f\Big(\frac{x-\eta_0}{L},\overline{\phi}(x-\eta(t)-\overline{x}L)\Big)\geq-\overline{c}\,\beta-\frac{C_2C_1^{\alpha}}{L^{\alpha}},$$

for some positive number C_2 independent of L, u and (t,x). Thus, even if it means increasing L^* (but still independently of u and (t,x)), one has $N_2(t,x)\geq0$ for all $L>L^*$ and $(t,x)\in(0,+\infty)\times\mathbb{R}$ with $|x-\eta(t)-L\overline{x}|\leq C$ and $v_L(t,x)<1$. Combining the above properties, one has $N_2(t,x)\geq0$ for all $L>L^*$ and $(t,x)\in(0,+\infty)\times\mathbb{R}$ with $v_L(t,x)<1$.

By choosing $q(t)$ and $\eta(t)$ as in (4.14), we then get the inequality (4.11) and $N(t,x)\geq0$ for all $(t,x)\in(0,+\infty)\times\mathbb{R}$ such that $v_L(t,x)<1$. Since $\xi_s\in L\mathbb{Z}$ for all $s\in\mathbb{R}$ and since $u<1$ in $\mathbb{R}^2$, the maximum principle implies that

$$u(s+t,x+\xi_s)\leq\min\big(\overline{u}_L(x-\eta(t))+q(t),1\big)\ \text{ for all } t\geq0,\ s\in\mathbb{R}\text{ and }x\in\mathbb{R}.$$

Set $x^*=\sup_{t\geq0}\eta(t)=\eta_0+q_0(2K+\gamma)/(\gamma\rho)$ and $\gamma_0=\gamma/2$. Since $\overline{u}_L$ is decreasing in $\mathbb{R}$, the second inequality of (4.10) follows immediately.

As already emphasized, the proof of the first inequality follows the same scheme. Hence, the proof of Lemma 4.5 is complete. □

4.2 Proof of Theorem 1.7

Based on the above preparations, we are in a position to carry out the proof of Theorem 1.7. Throughout the proof, we fix a period $L>L^*$, where $L^*>0$ is given in Lemmas 4.3 and 4.5. The proof will be divided into five main steps.

Step 1: some useful notations. Let $E=BUC(\mathbb{R},\mathbb{R})$ be the Banach space of bounded uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}$, endowed with the L^∞ norm $\|\cdot\|_E=\|\cdot\|_{L^\infty(\mathbb{R})}$ and the usual order in $C(\mathbb{R},\mathbb{R})$, that is, $u\leq_E v$ if $u(x)\leq v(x)$ for all $x\in\mathbb{R}$ and $u<_E v$ if $u(x)\leq,\not\equiv v(x)$ in $\mathbb{R}$. For $u\in E$ and $\sigma>0$, $B_E(u,\sigma)=\big\{v\in E\ ;\ \|v-u\|_E<\sigma\big\}$ is the open ball with center u and radius σ. For any

$u_0 \in E$, let $(t, x) \mapsto u(t, x; u_0)$ denote the unique solution of (1.1) in $(0, +\infty) \times \mathbb{R}$ with initial value $u(0, \cdot; u_0) = u_0$. This solution exists for all $t > 0$ since $\partial_u f$ is bounded in $\mathbb{R}^2$. For any $t \geq 0$, set

$$S_t[u_0] = u(t, \cdot; u_0).$$

Then $(S_t)_{t>0}$ is a continuous semiflow on E, and the strong parabolic maximum principle implies that $(S_t)_{t>0}$ is strictly order-preserving on E in the sense that if u_0 and v_0 belong to E and $u_0 \leq, \not\equiv v_0$ in $\mathbb{R}$, then $S_t[u_0] < S_t[v_0]$ in $\mathbb{R}$ for every $t > 0$. An equilibrium point $p \in E$ for the semiflow $(S_t)_{t>0}$ is nothing but a stationary solution of (1.1). Lastly, given two equilibrium points $p_\pm$ in E, an entire orbit connecting p_- to p_+ is a continuous map $\Gamma : \mathbb{R} \to E$ such that $\Gamma(t) = S_{t-s}[\Gamma(s)]$ for all $s < t$ in $\mathbb{R}$ and $\Gamma(t) \to p_\pm$ in E as $t \to \pm\infty$.

Step 2: the existence of some ordered stationary fronts. From Lemmas 4.1 and 4.2, $0 < \underline{u}_L < 1$ is a strict subsolution and $0 < \overline{u}_L < 1$ is a strict subsolution of (1.1). Hence, from the parabolic maximum principle, $u(t, x; \underline{u}_L)$ is increasing in $t > 0$, while $u(t, x; \overline{u}_L)$ is decreasing in $t > 0$. From standard parabolic estimates, $u(t, x; \underline{u}_L)$ converges as $t \to +\infty$ to a stationary solution $0 \leq u_1(x) \leq 1$ of (1.1) and $u(t, x; \overline{u}_L)$ converges as $t \to +\infty$ to a stationary solution $0 \leq u_2(x) \leq 1$ of (1.1), locally uniformly in $x \in \mathbb{R}$. Furthermore, since $0 < \underline{u}_L < \overline{u}_L < 1$ in $\mathbb{R}$ from Lemma 4.3, there holds

$$0 < \underline{u}_L(x) < u(t, x; \underline{u}_L) < u(t, x; \overline{u}_L) < \overline{u}_L(x) < 1 \text{ for all } x \in \mathbb{R} \text{ and } t > 0,$$

from the strong maximum principle, whence

$$0 < \underline{u}_L < u_1 \leq u_2 < \overline{u}_L < 1 \text{ in } \mathbb{R}. \tag{4.15}$$

On the other hand, since $\overline{u}_L(-\infty) = \underline{u}_L(-\infty) = 1$ and $\overline{u}_L(+\infty) = \underline{u}_L(+\infty) = 0$, one has $u_1(-\infty) = u_2(-\infty) = 1$, $u_1(+\infty) = u_2(+\infty) = 0$, while $u(t, -\infty; \overline{u}_L) = u(t, -\infty; \underline{u}_L) = 1$ and $u(t, +\infty; \overline{u}_L) = u(t, +\infty; \underline{u}_L) = 0$ uniformly in $t > 0$. Therefore, $u(t, \cdot; \underline{u}_L) \to u_1$ and $u(t, \cdot; \overline{u}_L) \to u_2$ as $t \to +\infty$ uniformly in $\mathbb{R}$, that is, $S_t[\underline{u}_L] \to u_1$ and $S_t[\overline{u}_L] \to u_2$ in E as $t \to +\infty$.

The functions u_1 and u_2 are thus ordered stationary fronts connecting 0 and 1 for (1.1). The lower front u_1 is approached from below by a solution of the Cauchy problem associated to (1.1), while the upper front u_2 is approached from above. For the conclusion of Theorem 1.7, we would need the opposite situation, namely a lower front which is approached from above and an upper front which is approached from below. To do so, we will, say, keep u_2 as such and shift u_1 to the right in order to make it larger than u_2. Namely, since both u_1 and u_2 have the same exponential convergence rates $\lambda_\pm$ at $\pm\infty$ in the sense of (4.9) in Lemma 4.4, and since both u_1 and u_2 are continuous and range in $(0, 1)$, it is elementary to check that there is $k \in \mathbb{N}$ such that

$$0 < u_2(x) < u_3(x) := u_1(x - kL) < 1 \text{ for all } x \in \mathbb{R}. \tag{4.16}$$

Step 3: the stability from above and below of u_2 and u_3. Let us define the order interval

$$I := [u_2, u_3]_E = \{u \in E \ ; \ u_2 \leq_E u \leq_E u_3\} = \{u \in BUC(\mathbb{R}, \mathbb{R}) \ ; \ u_2 \leq u \leq u_3 \text{ in } \mathbb{R}\}$$

and let us show in this step that u_2 is stable from above in I for the semiflow $(S_t)_{t>0}$ and that u_3 is stable from below in I for the semiflow $(S_t)_{t>0}$, in the sense that there is $\sigma > 0$ such that

$$\begin{cases} \forall\, u \in I \cap B_E(u_2, \sigma), & S_t[u] \to u_2 \text{ in } E \text{ as } t \to +\infty, \\ \forall\, u \in I \cap B_E(u_3, \sigma), & S_t[u] \to u_3 \text{ in } E \text{ as } t \to +\infty. \end{cases}$$

Indeed, first of all, it follows from (4.15) and (4.16) that $u_2 < \min(u_3, \overline{u}_L) \le u_3$ in $\mathbb{R}$. Observe from (4.2), (4.9) and the negativity of $\overline{c}$ that $0 < \overline{\lambda}_+ < \lambda_+$, $0 < \lambda_- < \overline{\lambda}_-$ and that

$$\lim_{x\to+\infty} \frac{u_3(x)}{\overline{u}_L(x)} = 0 \ \text{ and } \ \lim_{x\to-\infty} \frac{1-u_3(x)}{1-\overline{u}_L(x)} = +\infty.$$

Therefore, there is $C > 0$ such that $u_3(x) < \overline{u}_L(x)$ for all $|x| \ge C$, whence $\min(u_3, \overline{u}_L) = u_3$ in $\mathbb{R}\backslash(-C, C)$. On the other hand, by continuity of u_2, u_3 and $\overline{u}_L$ and the inequality $u_2 < \min(u_3, \overline{u}_L)$ in $\mathbb{R}$, there is $\sigma > 0$ such that $u_2(x) + \sigma \le \min(u_3(x), \overline{u}_L(x))$ for all $|x| \le C$. As a consequence, for any $u \in I \cap B_E(u_2, \sigma) = [u_2, u_3]_E \cap B_E(u_2, \sigma)$, one has

$$\begin{cases} u_2(x) \le u(x) \le u_3(x) = \min(u_3(x), \overline{u}_L(x)) & \text{for all } |x| \ge C, \\ u_2(x) \le u(x) < u_2(x) + \sigma \le \min(u_3(x), \overline{u}_L(x)) & \text{for all } |x| \le C, \end{cases} \tag{4.17}$$

whence $u_2 \le u \le \min(u_3, \overline{u}_L) \le \overline{u}_L$ in $\mathbb{R}$. Since u_2 is a stationary solution of (1.1) and since $S_t[\overline{u}_L] \to u_2$ in E as $t \to +\infty$, one concludes from the maximum principle that $S_t[u] \to u_2$ in E as $t \to +\infty$ for every $u \in I \cap B_E(u_2, \sigma)$, namely u_2 is stable from above in I for the semiflow $(S_t)_{t>0}$.

Similarly, it follows from (4.8) and (4.9) that $0 < \lambda_+ < \underline{\lambda}_+$, $0 < \underline{\lambda}_- < \lambda_-$ and that $\max(u_2(x), \underline{u}_L(x - kL)) = u_2(x)$ for all $|x| \ge C'$, for some $C' > 0$. On the other hand, $\underline{u}_L(\cdot - kL) \le \max(u_2, \underline{u}_L(\cdot - kL)) < u_3$ in $\mathbb{R}$ from (4.15) and (4.16). Hence, as in the previous paragraph, there is $\sigma' > 0$ such that, for all $u \in I \cap B_E(u_3, \sigma')$, there holds

$$\underline{u}_L(\cdot - kL) \le \max(u_2, \underline{u}_L(\cdot - kL)) \le u \le u_3 \ \text{ in } \mathbb{R},$$

whence $S_t[u] \to u_3$ in E as $t \to +\infty$ (notice that $S_t[\underline{u}_L(\cdot - kL)] = S_t[\underline{u}_L](\cdot - kL) \to u_1(\cdot - kL) = u_3$ in E as $t \to +\infty$, since (1.1) is L-periodic in x). Therefore, u_3 is stable from below in I for the semiflow $(S_t)_{t>0}$.

Step 4: the existence of transition fronts which are not pulsating fronts. From the previous steps, $u_2 < u_3$ are ordered equilibria for the continuous and strictly order-preserving semiflow $(S_t)_{t>0}$, and u_2 and u_3 are stable respectively from above and below in $I = [u_2, u_3]_E$ for $(S_t)_{t>0}$. Furthermore, for any $t > 0$, the set $S_t[I]$ is precompact with respect to the compact open topology, from standard parabolic estimates. Since $u_2(-\infty) = u_3(-\infty) = 1$ and $u_2(+\infty) = u_3(+\infty) = 0$, and since the semiflow is order-preserving, it follows then that $S_t[I]$ is precompact in E with respect to the L^∞-norm for every $t > 0$.

It follows then from the Dancer-Hess connecting orbit lemma (see, e.g., [27, Proposition 9.1]) that the semiflow $(S_t)_{t>0}$ has an equilibrium u^* in $I\backslash\{u_2, u_3\}$. From the strong elliptic maximum principle, u^* is a stationary solution of (1.1) such that $u_2 < u^* < u_3$ in $\mathbb{R}$.

Let us now define the set

$$\mathcal{E} = \big\{p \in [u_2, u^*]_E\backslash\{u_2\} \ ; \ p \text{ is an equilibrium of } (S_t)_{t>0}\big\},$$

that is, $\mathcal{E}$ is the set of stationary solutions p of (1.1) such that $u_2 \le, \not\equiv p \le u^*$ in $\mathbb{R}$. We will apply Zorn lemma to show the existence of a minimal element in $\mathcal{E}$. First of all, notice that $\mathcal{E}$ is not empty, since $u^* \in \mathcal{E}$. Consider now any non-empty and totally ordered subset $\mathcal{F}$ of $\mathcal{E}$ and let us show that there is $p \in \mathcal{E}$ such that $p \le_E q$ for all $q \in \mathcal{F}$. Namely, define

$$p(x) = \inf\big\{q(x) \ ; \ q \in \mathcal{F}\big\}.$$

One immediately has

$$0 < u_2 \leq p \leq q \leq u^* \leq u_3 < 1 \text{ in } \mathbb{R} \text{ for all } q \in \mathcal{F}.$$

In particular, $p \in [u_2, u^*]_E$. One shall show that $p \in \mathcal{E}$, that is, $p \not\equiv u_2$ in $\mathbb{R}$ and p is a stationary solution of (1.1).

First of all, we claim that, for any $q \in \mathcal{F}$, there is $x_q \in \mathbb{R}$ such that

$$q(x_q) > \min(u_3(x_q), \overline{u}_L(x_q)).$$

Otherwise, $(u_2 \leq)\, q \leq \min(u_3, \overline{u}_L)\, (\leq \overline{u}_L)$ in $\mathbb{R}$ and $q = S_t[q] \to u_2$ in E as $t \to +\infty$, whence $q = u_2$, contradicting $q \in \mathcal{E}$. Thus, $q(x_q) > \min(u_3(x_q), \overline{u}_L(x_q))$ for some $x_q \in \mathbb{R}$. Furthermore, $q \leq u^* \leq u_3$ in $\mathbb{R}$ and $\min(u_3(x), \overline{u}_L(x)) = u_3(x)$ for all $|x| \geq C$, where $C > 0$ independent of q is as in (4.17). Therefore $|x_q| < C$, whence $q(x_q) > \min(u_3(x_q), \overline{u}_L(x_q)) > u_2(x_q) + \sigma$, where $\sigma > 0$ independent of q is as in (4.17). Since $q \geq u_2$ are two stationary solutions of (1.1), it follows then from Harnack inequality that there is a constant $\omega > 0$, independent of q, such that $\min_{[-C,C]}(q - u_2) \geq \omega > 0$. As a consequence, $p(x) \geq u_2(x) + \omega$ for all $|x| \leq C$. In particular, $p \not\equiv u_2$ in $\mathbb{R}$.

Let us then show that p is a stationary solution of (1.1). First of all, since $\mathcal{F}$ is totally ordered and consists of stationary solutions of (1.1), the strong maximum principle implies that, for any $q, \widetilde{q} \in \mathcal{F}$, there holds either $q < \widetilde{q}$ in $\mathbb{R}$, or $q > \widetilde{q}$ in $\mathbb{R}$, or $q = \widetilde{q}$ in $\mathbb{R}$. By definition of p, there is a sequence $(p_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ such that

$$p_n(0) \to p(0) \text{ as } n \to +\infty.$$

Since $p_n(0) \geq p(0)$ for all $n \in \mathbb{N}$, one can assume without loss of generality that $p_n(0) \geq p_{n+1}(0)$ for all $n \in \mathbb{N}$, whence $p_n \geq p_{n+1}$ in $\mathbb{R}$ for all $n \in \mathbb{N}$. Since $q(0) \geq p(0)$ for all $q \in \mathcal{F}$ by definition of p, two cases may then occur:

$$\text{either there is } q_0 \in \mathcal{F} \text{ such that } q_0(0) = p(0), \quad \text{or } q(0) > p(0) \text{ for all } q \in \mathcal{F}.$$

In the first case, one infers that $q_0(0) = p(0) \leq q(0)$ for all $q \in \mathcal{F}$, whence $(p \leq)\, q_0 \leq q$ in $\mathbb{R}$ for all $q \in \mathcal{F}$; finally $q_0 = p$ and thus $p \in \mathcal{E}$. In the second case, one has $q(0) > p(0)$ for all $q \in \mathcal{F}$. Fix any $x \in \mathbb{R}$ and $\varepsilon > 0$. By definition of p, there is $q \in \mathcal{F}$ such that $p(x) \leq q(x) \leq p(x) + \varepsilon$. Since $p(0) < q(0)$, there is $n_0 \in \mathbb{N}$ such that $p(0) \leq p_n(0) < q(0)$ for all $n \geq n_0$, whence $p_n < q$ in $\mathbb{R}$ and $p(x) \leq p_n(x) < q(x) \leq p(x) + \varepsilon$ for all $n \geq n_0$. Therefore, $p_n(x) \to p(x)$ as $n \to +\infty$ for all $x \in \mathbb{R}$. Since $0 < u_2 \leq p_n \leq u^* < u_3 < 1$ in $\mathbb{R}$ and each p_n is a stationary solution of (1.1), standard elliptic estimates imply then that $p_n \to p$ in $C^2_{loc}(\mathbb{R})$ and p is thus a stationary solution of (1.1).

Finally, in all cases, one has shown that $p \in [u_2, u^*]_E \backslash \{u_2\}$ is an equilibrium of the semiflow $(S_t)_{t>0}$ such that $p \leq q$ in $\mathbb{R}$ for all $q \in \mathcal{F}$. In other words, p is a minorant of $\mathcal{F}$ which belongs to $\mathcal{E}$. Therefore, Zorn lemma provides the existence of a minimal element u_4 in $\mathcal{E}$, the minimality meaning that there is no element $p \in \mathcal{E}\backslash\{u_4\}$ such that $p \leq u_4$. In other words, u_4 is a stationary solution of (1.1) such that $u_2 < u_4 \leq u^*$ (from the strong maximum principle) and the semiflow $(S_t)_{t>0}$ has no equilibrium in $[u_2, u_4]_E \backslash \{u_2, u_4\}$. It follows then from the Dancer-Hess connecting orbit lemma and from the stability of u_2 from above in I that there is a time-decreasing entire orbit connecting u_4 to u_2, that is, a solution $u : \mathbb{R}^2 \to \mathbb{R}$ of (1.1) such that $u(t,x)$ is time-decreasing, with $u(t,x) \to u_4(x)$ as $t \to -\infty$ and $u(t,x) \to u_2(x)$ as $t \to +\infty$ uniformly in $x \in \mathbb{R}$. In particular,

$$0 < u_2(x) < u(t,x) < u_4(x) \leq u^*(x) < u_3(x) < 1 \text{ for all } (t,x) \in \mathbb{R}^2$$

and since $u_2(-\infty) = u_3(-\infty) = 1$ and $u_2(-\infty) = u_3(-\infty) = 0$, one gets that u is a transition front connecting 0 and 1 for (1.1), with for instance $\xi_t = 0$ for all $t \in \mathbb{R}$.

Similarly, one can show the existence of a stationary solution u_5 and a time-increasing solution $v(t, x)$ of (1.1) such that

$$0 < u_2(x) < u^*(x) \leq u_5(x) < v(t, x) < u_3(x) < 1 \quad \text{for all } (t, x) \in \mathbb{R}^2,$$

with $v(t, x) \to u_5(x)$ as $t \to -\infty$ and $v(t, x) \to u_3(x)$ as $t \to +\infty$ uniformly in $x \in \mathbb{R}$. Furthermore, v is a transition front connecting 0 and 1 for (1.1). Finally, $0 < u_2 < u_4 \leq u^* \leq u_5 < u_3$ in $\mathbb{R}$ and the desired conclusion of Theorem 1.7 follows with $u^- = u_2$, $u^+ = u_4$, $v^- = u_5$ and $v^+ = u_3$.

Step 5: the characterization of global mean speeds. Lastly, we show that for any transition front $\widetilde{u}$ connecting 0 and 1 for (1.1) and associated with positions $(\xi_t)_{t\in\mathbb{R}}$, there is $M \geq 0$ such that $|\xi_t - \xi_s| \leq M$ for all $(t, s) \in \mathbb{R}^2$. Assume not. By Lemma 2.1, there are then two sequences $(t_k)_{k\in\mathbb{N}}$ and $(s_k)_{k\in\mathbb{N}}$ of real numbers such that $|t_k - s_k| \to +\infty$ and $|\xi_{t_k} - \xi_{s_k}| \to +\infty$ as $k \to +\infty$. Without loss of generality, one can assume that $s_k < t_k$ for all $k \in \mathbb{N}$.

Suppose firstly that, up to extraction of some subsequence, $\xi_{t_k} - \xi_{s_k} \to +\infty$ as $k \to +\infty$. From Definition 1.1, one then infers that, for any $x \in \mathbb{R}$, $\lim_{k\to+\infty} \widetilde{u}(t_k, x + \xi_{s_k}) = 1$. On the other hand, it follows from Lemma 4.5, there are $x^* \in \mathbb{R}$, $\gamma_0 > 0$ and $q_0 > 0$ such that

$$\widetilde{u}(t_k, x + \xi_{s_k}) \leq \min\big(\overline{u}_L(x - x^*) + q_0 e^{-\gamma_0(t_k - s_k)}, 1\big) \quad \text{for all } k \in \mathbb{N} \text{ and } x \in \mathbb{R}.$$

Notice that, for any $x \in \mathbb{R}$, $\lim_{k\to+\infty}(\overline{u}_L(x - x^*) + q_0 e^{-\gamma_0(t_k - s_k)}) = \overline{u}_L(x - x^*) < 1$. Therefore, for every $x \in \mathbb{R}$, $\limsup_{k\to+\infty} \widetilde{u}(t_k, x + \xi_{s_k}) < 1$. One has then reached a contradiction. Secondly, in the case where, up to extraction of some subsequence, $\xi_{t_k} - \xi_{s_k} \to -\infty$ as $k \to +\infty$, one can derive a similar contradiction by using the lower bound for $\widetilde{u}$ in Lemma 4.5.

Therefore, there is $M > 0$ such that $|\xi_t - \xi_s| \leq M$ for all $(t, s) \in \mathbb{R}^2$. In particular, every transition front connecting 0 and 1 for (1.1) has zero global mean speed. The proof of Theorem 1.7 is thereby complete. $\square$

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